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SUPPLEMENT OF "OPTIMAL ADAPTIVITY OF SIGNED-POLYGON STATISTICS FOR NETWORK TESTING"

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This supplement contains additional results and technical proofs for the main article [4]. Appendix A studies the behavior of the SgnT test statistic and proves Theorems 2.1, 2.3, and 2.5. Appendix B is about the properties of the SgnQ test statistic and proves Theorems 2.2, 2.4, and 2.6. Appendix C derives the matrix forms of signed-polygon statistics and proves Theorem 1.1. Appendix D studies the estimation error of $\|\theta\|^2$ and proves Lemma 2.1. Appendix E contains spectral analysis for Ω and $\widetilde{\Omega}$ and proves Lemmas 2.2-2.3. Appendix F analyzes the region of impossibility and proves Lemma 3.1 and Theorems 3.1-3.5. Appendix G calculates the mean and variance of signed-polygon statistics and proves the results in Tables 1-2, Tables A.1-A.2, Theorems 4.1-4.3, and Theorems A.1-A.3. Appendix H contains additional simulation results.

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APPENDIX A: THE BEHAVIOR OF THE SGNT TEST STATISTIC

We now discuss the behavior of the SgnT test statistic and prove Theorems 2.1, 2.3, and 2.5. The discussion is similar to that of SgnQ in Section 4, and so we keep it brief.

Recall that the SgnT test statistic is defined by

$$T_n = \sum_{i_1,i_2,i_3(dist)} (A_{i_1i_2} - \hat{\eta}_{i_1}\hat{\eta}_{i_2})(A_{i_2i_3} - \eta_{i_2}\hat{\eta}_{i_3})(A_{i_3i_1} - \hat{\eta}_{i_3}\hat{\eta}_{i_1}).$$

Similarly, define the Ideal SgnT test statistic $\widetilde{T}_n$ and the Proxy SgnT test statistic and T_n^*, and write

$$T_n = \widetilde{Q}_n + (Q_n^* - \widetilde{Q}_n) + (Q_n - Q_n^*). \tag{1}$$

We have the following observations.

- $\widetilde{Q}_n$ is the sum of 8 different post-expansion sums, divided into 4 types. See Table A.1.
- $Q_n^* - \widetilde{Q}_n$ is the sum of 19 different post-expansion sums, divided into 6 different types. See Table A.2.
- $Q_n - Q_n^*$ is the sum of 37 different post-expansion sums.

The following lemmas are proved in the supplementary material.

TABLE A.1
The 4 types of the 8 post-expansion sums for $\widetilde{T}_n$ ($\|\theta\|_q$ is the ℓ^q-norm of θ (the subscript is dropped when $q = 2$). In our setting, $\alpha\|\theta\| \to \infty$, and $\|\theta\|_4^4 \ll \|\theta\|_3^3 \ll \|\theta\|^2 \ll \|\theta\|_1$.

Type	#	$(N_{\widetilde{\Omega}}, N_W)$	Examples	Mean	Variance
I	1	(0, 3)	$\sum_{i,j,k(dist)} W_{ij}W_{jk}W_{ki}$	0	$\asymp \|\theta\|^6$
II	3	(1, 2)	$\sum_{i,j,k(dist)} \widetilde{\Omega}_{ij}W_{jk}W_{ki}$	0	$\le C\alpha^2\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^6)$
III	3	(2, 1)	$\sum_{i,j,k(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}W_{ki}$	0	$\le C\alpha^4\|\theta\|^4\|\theta\|_3^6$
IV	1	(3, 0)	$\sum_{i,j,k(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{ki}$	$\sim \mathrm{tr}(\widetilde{\Omega}^3)$	0

THEOREM A.1 (Ideal SgnT test statistic). *Consider the testing problem (1.6) under the DCMM model (1.1)-(1.4), where the condition (2.2) is satisfied under the alternative hypothesis. Suppose $\theta_{\max} \to 0$ and $\|\theta\| \to \infty$ as $n \to \infty$, and suppose $|\lambda_2|/\sqrt{\lambda_1} \to \infty$ under the alternative hypothesis. Then, under the null hypothesis, as $n \to \infty$,*

$$\mathbb{E}[\widetilde{T}_n] = 0, \qquad \mathrm{Var}(\widetilde{T}_n) = 6\|\theta\|^6 \cdot [1 + o(1)],$$

and

$$\frac{\widetilde{T}_n - \mathbb{E}[\widetilde{T}_n]}{\sqrt{\mathrm{Var}(\widetilde{T}_n)}} \longrightarrow N(0,1), \qquad \textit{in law}.$$

Furthermore, under the alternative hypothesis, as $n \to \infty$,

$$\mathbb{E}[\widetilde{T}_n] = \mathrm{tr}(\widetilde{\Omega}^3) + o(\|\theta\|^3), \qquad \mathrm{Var}(\widetilde{T}_n) \le C\|\theta\|^6 + C(|\lambda_2|/\lambda_1)^4\|\theta\|^4\|\theta\|_3^6.$$

THEOREM A.2 (Proxy SgnT test statistic). *Consider the testing problem (1.6) under the DCMM model (1.1)-(1.4), where the condition (2.2) is satisfied under the alternative hypothesis. Suppose $\theta_{\max} \to 0$ and $\|\theta\| \to \infty$ as $n \to \infty$, and suppose $|\lambda_2|/\sqrt{\lambda_1} \to \infty$ under the alternative hypothesis. Then, under the null hypothesis, as $n \to \infty$,*

$$\mathbb{E}[T_n^* - \widetilde{T}_n] = o(\|\theta\|^3), \qquad \mathrm{Var}(T_n^* - \widetilde{T}_n) = o(\|\theta\|^6).$$

TABLE A.2

The 6 types of the 19 post-expansion sums for $(T_n^* - \widetilde{T}_n)$. Notations: same as Table A.1.

Type	#	$(N_\delta, N_{\widetilde{\Omega}}, N_W)$	Examples	Abs. Mean	Variance
Ia	3	(1, 0, 2)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} W_{jk} W_{ki}$	0	$\leq C\frac{\|\theta\|^4\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^6)$
Ib	6	(1, 1, 1)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} \widetilde{\Omega}_{jk} W_{ki}$	$\leq C\alpha\|\theta\|^4 = o(\alpha^3\|\theta\|^6)$	$\leq \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^6)$
Ic	3	(1, 2, 0)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} \widetilde{\Omega}_{jk} \widetilde{\Omega}_{ki}$	0	$\leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1} = O(\alpha^4\|\theta\|^4\|\theta\|_3^6)$
IIa	3	(2, 0, 1)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} \delta_{jk} W_{ki}$	$\leq C\|\theta\|^2 = o(\|\theta\|^3)$	$\leq C\|\theta\|_3^6 = o(\|\theta\|^6)$
IIb	3	(2, 1, 0)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} \delta_{jk} \widetilde{\Omega}_{ki}$	$\leq \frac{C\alpha\|\theta\|^6}{\|\theta\|_1^2} = o(\|\theta\|^3)$	$\leq \frac{C\alpha^2\|\theta\|^{10}}{\|\theta\|_1^2} = o(\|\theta\|^6)$
III	1	(3, 0, 0)	$\sum_{\substack{i,j,k \\ (dist)}} \delta_{ij} \delta_{jk} \delta_{ki}$	$\leq \frac{C\|\theta\|^4}{\|\theta\|_1^2} = o(\|\theta\|^3)$	$\leq \frac{C\|\theta\|^4\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^6)$

Furthermore, under the alternative hypothesis,

$$\mathbb{E}[T_n^* - \widetilde{T}_n] = o((|\lambda_2|/\lambda_1)^3\|\theta\|^6),$$
$$\mathrm{Var}(T_n^* - \widetilde{T}_n) \leq C(|\lambda_2|/\lambda_1)^4\|\theta\|^4\|\theta\|_3^6 + o(\|\theta\|^6).$$

THEOREM A.3 (Real SgnT test statistic). *Consider the testing problem (1.6) under the DCMM model (1.1)-(1.4), where the condition (2.2) is satisfied under the alternative hypothesis. Suppose $\theta_{\max} \to 0$ and $\|\theta\| \to \infty$ as $n \to \infty$, and suppose $|\lambda_2|/\sqrt{\lambda_1} \to \infty$ under the alternative hypothesis. Then, under the null hypothesis, as $n \to \infty$,*

$$|\mathbb{E}[T_n - T_n^*]| = o(\|\theta\|^3), \qquad \text{and} \qquad \mathrm{Var}(T_n - T_n^*) = o(|\theta\|^6).$$

Under the alternative hypothesis, as $n \to \infty$,

$$|\mathbb{E}[T_n - T_n^*]| = o((|\lambda_2|/\lambda_1)^3\|\theta\|^6),$$
$$\mathrm{Var}(T_n - T_n^*) = o((|\lambda_2|/\lambda_1)^4\|\theta\|^4|\theta\|_3^6) + o(\|\theta\|^6).$$

Combining Theorems A.1, A.2, and A.3, Theorems 2.1, 2.3, and 2.5 follow by similar arguments as in Appendix B.

APPENDIX B: THE BEHAVIOR OF THE SGNQ TEST STATISTIC

We prove Theorems 2.2, 2.4, and 2.6. We use the same notations as those in Section 4 of the main article, and the proof here relies on Theorems 4.1-4.3 in the main article.

Consider Theorem 2.2. In this theorem, we assume the null is true. First, by Theorems 4.2 and 4.3 and elementary statistics, $\mathbb{E}[Q_n^* - \widetilde{Q}_n] \sim 2\|\theta\|^4$, $|\mathbb{E}[Q_n - Q_n^*]| = o(\|\theta\|^4)$, $\mathrm{Var}(Q_n^* - \widetilde{Q}_n) = o(\|\theta\|^8)$, and $\mathrm{Var}(Q_n - Q_n^*) = o(\|\theta\|^8)$. It follows that

(2) $$\mathbb{E}[Q_n] - \mathbb{E}[\widetilde{Q}_n] = (2 + o(1))\|\theta\|^4, \qquad \mathrm{Var}(Q_n - \widetilde{Q}_n) = o(\|\theta\|^8).$$

By Theorem 4.1.

(3) $$\mathbb{E}[\widetilde{Q}_n] = o(\|\theta\|^4), \quad \mathrm{Var}(\widetilde{Q}_n) \sim 8\|\theta\|^8, \quad \frac{\widetilde{Q}_n - \mathbb{E}[\widetilde{Q}_n]}{\sqrt{\mathrm{Var}(\widetilde{Q}_n)}} \to N(0,1).$$

Since for any random variables X and Y, $\mathrm{Var}(X+Y) \leq (1+a_n)\mathrm{Var}(X) + (1+\frac{1}{a_n})\mathrm{Var}(Y)$ for any number $a_n > 0$, combining the above and letting a_n tend to 0 appropriately slow,

(4) $$\mathbb{E}[Q_n] \sim 2\|\theta\|^4, \qquad \mathrm{Var}(Q_n) \sim 8\|\theta\|^8.$$

Moreover, write

$$\frac{Q_n - \mathbb{E}[Q_n]}{\sqrt{\mathrm{Var}(Q_n)}} = \sqrt{\frac{\mathrm{Var}(\widetilde{Q}_n)}{\mathrm{Var}(Q_n)}} \cdot \Big[\frac{(Q_n - \widetilde{Q}_n)}{\sqrt{\mathrm{Var}(\widetilde{Q}_n)}} + \frac{\widetilde{Q}_n - \mathbb{E}[\widetilde{Q}_n]}{\sqrt{\mathrm{Var}(\widetilde{Q}_n)}} + \frac{\mathbb{E}[\widetilde{Q}_n] - \mathbb{E}[Q_n]}{\sqrt{\mathrm{Var}(\widetilde{Q}_n)}}\Big].$$

On the right hand side, by (2)-(4), as $n \to \infty$, the term outside the bracket $\to 1$, and for the three terms in the bracket, the first one has a mean and variance that tend to 0 so it tends to 0 in probability, the second one weakly converges to $N(0,1)$, and the last one $\to 0$. Combining these,

$$\frac{Q_n - \mathbb{E}[Q_n]}{\sqrt{\mathrm{Var}(Q_n)}} \to N(0,1), \qquad \text{in law.} \tag{5}$$

Combining (4) and (5) proves Theorem 2.2.

Next, we consider Theorem 2.4, where we assume the alternative is true. First, similarly, by Theorems 4.2 and 4.3,

$$\mathbb{E}[Q_n^* - \widetilde{Q}_n] = (2 + o(1))\|\theta\|^4 + o((|\lambda_2|/\lambda_1)^4\|\theta\|^8),$$
$$\mathrm{Var}(Q_n - \widetilde{Q}_n) \le C(\lambda_2/\lambda_1)^6\|\theta\|^8\|\theta\|_3^6 + o(\|\theta\|^8).$$

Second, by Theorem 4.1,

$$\mathbb{E}[\widetilde{Q}_n] = \mathrm{tr}(\widetilde{\Omega}^4) + o(\|\theta\|^4), \qquad \mathrm{Var}(\widetilde{Q}_n) \le C[\|\theta\|^8 + (\lambda_2/\lambda_1)^6\|\theta\|^8\|\theta\|_3^6].$$

Combining these proves Theorem 2.4.

Last, we consider Theorems 2.5-2.6. Since the proofs are similar, we only show Theorem 2.6. First, by Theorem 2.2 and Lemma 2.1, under the null, $\frac{Q_n - 2(\|\hat{\eta}\|^2-1)^2}{\sqrt{8(\|\hat{\eta}\|^2-1)^4}} \to N(0,1)$, so the Type I error is

$$\mathbb{P}_{H_0^{(n)}}\Big(Q_n \ge (2 + z_\alpha\sqrt{8})(|\hat{\eta}\|^2 - 1)^2\Big) = P\Big(\frac{Q_n - 2(\|\hat{\eta}\|^2 - 1)^2}{\sqrt{8(\|\hat{\eta}\|^2 - 1)^4}} \ge z_\alpha\Big) = \alpha + o(1).$$

Second, fixing $0 < \epsilon < 1$, let A_ϵ be the event $\{(\|\hat{\eta}\|^2 - 1) \le (1+\epsilon)\|\eta^*\|^2\}$. By Lemma 2.1 and definitions, on one hand, over the event A_ϵ, $(\|\hat{\eta}\|^2 - 1) \le (1+\epsilon)\|\eta^*\|^2 \le C\|\theta\|^2$, and on the other hand, $\mathbb{P}(A_\epsilon^c) = o(1)$. Therefore, the Type II error

$$\begin{aligned} &\mathbb{P}_{H_1^{(n)}}\Big(Q_n \le (2 + z_\alpha\sqrt{8})(\|\hat{\eta}\|^2 - 1)^2\Big) \\ \le &\mathbb{P}_{H_1^{(n)}}\Big(Q_n \le (2 + z_\alpha\sqrt{8})(\|\hat{\eta}\|^2 - 1)^2, A_\epsilon\Big) + \mathbb{P}(A_\epsilon^c) \\ \le &\mathbb{P}_{H_1^{(n)}}\Big(Q_n \le C(2 + z_\alpha\sqrt{8})\|\theta\|^4\Big) + o(1), \end{aligned}$$

where by Chebyshev's inequality, the first term in the last line

$$\le [\mathbb{E}(Q_n) - C(2 + z_\alpha\sqrt{8})\|\theta\|^4]^{-2} \cdot \mathrm{Var}(Q_n). \tag{6}$$

By Lemma D.2 of the supplementary material and our assumptions, $\lambda_1 \asymp \|\theta\|^2$, $|\lambda_2|/\sqrt{\lambda_1} \to \infty$, and $\|\theta\| \to \infty$. Using Lemma 2.3 $\mathbb{E}[Q_n] \ge C\lambda_2^4 \gg \lambda_1^2$, and it follows that $\mathbb{E}(Q_n) \gg C(2 + z_\alpha\sqrt{8})\|\theta\|^4$, so for sufficiently large n,

$$\mathbb{E}(Q_n) - C(2 + z_\alpha\sqrt{8})\|\theta\|^4 \ge \frac{1}{2}\mathbb{E}[Q_n] \ge C\lambda_2^4.$$

At the same time, by Theorem 2.4,

$$\mathrm{Var}(Q_n) \le C(\|\theta\|^8 + (\lambda_2/\lambda_1)^6\|\theta\|^8\|\theta\|_3^6).$$

Combining these, the right hand side of (6) does not exceed

$$C\frac{\|\theta\|^8 + (\lambda_2/\lambda_1)^6\|\theta\|^8\|\theta\|_3^6}{\lambda_2^8} = (I) + (II), \tag{7}$$

where $(I) = C\lambda_2^{-8}\|\theta\|^8$ and $(II) = C\lambda_2^{-8}(\lambda_2/\lambda_1)^6\|\theta\|^8\|\theta\|_3^6$. Now, first, since $\lambda_1 \asymp \|\theta\|^2$ and $|\lambda_2|/\sqrt{\lambda_1} \to 0$, $(I) \le C(\lambda_2/\sqrt{\lambda_1})^{-8} \to 0$. Second, since $\lambda_1 \asymp \|\theta\|^2$ and $\|\theta\|_3^6 \le \|\theta\|^4$, $(II) = C\lambda_2^{-2}\lambda_1^{-6}\|\theta\|^8\|\theta\|_3^6 \le C\lambda_2^{-2}$. As $|\lambda_2|/\sqrt{\lambda_1} \to \infty$, $\sqrt{\lambda_1} \asymp \|\theta\|$ with $\|\theta\| \to \infty$, $|\lambda_2| \to \infty$ and $(II) \to 0$. Inserting these into (7), the Type II error $\to 0$ and the claim follows. $\square$

APPENDIX C: MATRIX FORMS OF SIGNED-POLYGON STATISTICS

We prove Theorem 1.1. Recall that $\widetilde{A} = A - \hat{\eta}\hat{\eta}$. By definition,

$$T_n = \mathrm{tr}(\widetilde{A}^3) - \sum_{\substack{\text{at least two of}\\ i,j,k \text{ are equal}}} \widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki},$$

$$Q_n = \mathrm{tr}(\widetilde{A}^4) - \sum_{\substack{\text{at least two of}\\ i,j,k,\ell \text{ are equal}}} \widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{k\ell}\widetilde{A}_{\ell i}.$$

First, we derive the matrix form of T_n. If at least two of $\{i,j,k\}$ are equal, there are four cases: (a) $i = j$, $k \ne i$, (b) $j = k$, $i \ne j$, (c) $k = i$, $j \ne k$, (d) $i = j = k$. The first three cases are similar. It follows that

$$\begin{aligned} T_n &= \mathrm{tr}(\widetilde{A}^3) - 3\sum_{i,k(dist)} \widetilde{A}_{ii}\widetilde{A}_{ik}^2 - \sum_i \widetilde{A}_{ii}^3 \\ &= \mathrm{tr}(\widetilde{A}^3) - 3\Big(\sum_{i,k} \widetilde{A}_{ii}\widetilde{A}_{ik}^2 - \sum_i \widetilde{A}_{ii}^3\Big) - \sum_i \widetilde{A}_{ii}^3 \\ &= \mathrm{tr}(\widetilde{A}^3) - 3\mathrm{tr}(\widetilde{A} \circ \widetilde{A}^2) + 2\mathrm{tr}(\widetilde{A} \circ \widetilde{A} \circ \widetilde{A}). \end{aligned}$$

This gives the desired expression of T_n.

Next, we derive the matrix form of Q_n. When at least two of $\{i,j,k,\ell\}$ are equal, depending on how many indices are equal, we have four patterns: $\{i,i,i,i\}$, $\{i,i,i,j\}$, $\{i,i,j,j\}$, $\{i,i,j,k\}$, where (i,j,k) are distinct. For each pattern, depending on the appearing locations of the next distinct indices, there are a few variations. Take the pattern $\{i,i,j,k\}$ for example: (a) when a new distinct index appears at location 2 and at location 3, the variations are (i,j,k,i), (i,j,k,j), (i,j,k,k); (b) when a new distinct index appears at location 2 and at location 4, the variations are (i,j,i,k), (i,j,j,k); (c) when a new distinct index appears at location 3 and location 4, the variation is (i,i,j,k). Using similar arguments, we can find all variations of each pattern. They are summarized in Table C.3. Define

$$S_1 = \sum_{i,j,k(dist)} \widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki}, \qquad S_2 = \sum_{i,j,k(dist)} \widetilde{A}_{ij}^2\widetilde{A}_{ik}^2,$$

$$S_3 = \sum_{i,j(dist)} \widetilde{A}_{ii}^2\widetilde{A}_{ij}^2, \qquad S_4 = \sum_{i,j(dist)} \widetilde{A}_{ij}^4,$$

$$S_5 = \sum_{i,j(dist)} \widetilde{A}_{ii}\widetilde{A}_{ij}^2\widetilde{A}_{jj}, \qquad S_6 = \sum_i \widetilde{A}_{ii}^4.$$

TABLE C.3
Decomposition of $\mathrm{tr}(\widetilde{A}^4)$. *We note that the last column sums to* n^4.

Pattern	Variations	Summand	Sum	#Summands
$\{i,j,k,\ell\}$	(i,j,k,ℓ)	$\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{k\ell}\widetilde{A}_{\ell i}$	Q_n	$n(n-1)(n-2)(n-3)$
$\{i,i,j,k\}$	(i,j,k,i)	$\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki}\widetilde{A}_{ii}$	S_1	$6n(n-1)(n-2)$
	(i,j,k,j)	$\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{kj}\widetilde{A}_{ji}$	S_2	
	(i,j,k,k)	$\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{kk}\widetilde{A}_{ki}$	S_1	
	(i,j,i,k)	$\widetilde{A}_{ij}\widetilde{A}_{ji}\widetilde{A}_{ik}\widetilde{A}_{ki}$	S_2	
	(i,j,j,k)	$\widetilde{A}_{ij}\widetilde{A}_{jj}\widetilde{A}_{jk}\widetilde{A}_{ki}$	S_1	
	(i,i,j,k)	$\widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki}$	S_1	
$\{i,i,i,j\}$	(i,j,i,i)	$\widetilde{A}_{ij}\widetilde{A}_{ji}\widetilde{A}_{ii}\widetilde{A}_{ii}$	S_3	$4n(n-1)$
	(i,j,j,j)	$\widetilde{A}_{ij}\widetilde{A}_{jj}\widetilde{A}_{jj}\widetilde{A}_{ji}$	S_3	
	(i,i,j,i)	$\widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{ji}\widetilde{A}_{ii}$	S_3	
	(i,i,i,j)	$\widetilde{A}_{ii}\widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{ji}$	S_3	
$\{i,i,j,j\}$	(i,j,i,j)	$\widetilde{A}_{ij}\widetilde{A}_{ji}\widetilde{A}_{ij}\widetilde{A}_{ji}$	S_4	$3n(n-1)$
	(i,j,j,i)	$\widetilde{A}_{ij}\widetilde{A}_{jj}\widetilde{A}_{ji}\widetilde{A}_{ii}$	S_5	
	(i,i,j,j)	$\widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{jj}\widetilde{A}_{ji}$	S_5	
$\{i,i,i,i\}$	(i,i,i,i)	$\widetilde{A}_{ii}\widetilde{A}_{ii}\widetilde{A}_{ii}\widetilde{A}_{ii}$	S_6	n

It follows from Table C.3 that

$$Q_n = \mathrm{tr}(\widetilde{A}^4) - 4S_1 - 2S_2 - 4S_3 - S_4 - 2S_5 - S_6. \tag{8}$$

What remains is to derive the matrix form of S_1-S_6. By direct calculations,

$$\begin{aligned}
S_1 &= \sum_i \widetilde{A}_{ii}\Big[\sum_{j\neq i,k\neq i}\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki} - \sum_{j\neq i}\widetilde{A}_{ij}\widetilde{A}_{jj}\widetilde{A}_{ji}\Big]\\
&= \sum_i \widetilde{A}_{ii}\Big[\Big(\sum_{j,k}\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki} - 2\sum_j \widetilde{A}_{ij}^2\widetilde{A}_{ii} + \widetilde{A}_{ii}^3\Big) - \Big(\sum_j \widetilde{A}_{ij}^2\widetilde{A}_{jj} - \widetilde{A}_{ii}^3\Big)\Big]\\
&= \sum_{i,j,k}\widetilde{A}_{ii}\widetilde{A}_{ij}\widetilde{A}_{jk}\widetilde{A}_{ki} - 2\sum_{i,j}\widetilde{A}_{ii}^2\widetilde{A}_{ij}^2 - \sum_{i,j}\widetilde{A}_{ii}\widetilde{A}_{ij}^2\widetilde{A}_{jj} + 2\sum_i \widetilde{A}_{ii}^4\\
&= \mathrm{tr}(\widetilde{A}\circ\widetilde{A}^3) - 2\mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}^2) - 1_n'[\mathrm{diag}(\widetilde{A})(\widetilde{A}\circ\widetilde{A})\mathrm{diag}(\widetilde{A})]1_n + 2S_6.
\end{aligned}$$

Moreover, we can derive that

$$\begin{aligned}
S_2 &= \sum_i \Big[\sum_{j\neq i,k\neq i}\widetilde{A}_{ij}^2\widetilde{A}_{ik}^2 - \sum_{j\neq i}\widetilde{A}_{ij}^4\Big]\\
&= \sum_i \Big[\Big(\sum_{j,k}\widetilde{A}_{ij}^2\widetilde{A}_{ik}^2 - 2\sum_j \widetilde{A}_{ij}^2\widetilde{A}_{ii}^2 + \widetilde{A}_{ii}^4\Big) - \Big(\sum_j \widetilde{A}_{ij}^4 - \widetilde{A}_{ii}^4\Big)\Big]\\
&= \sum_{i,j,k}\widetilde{A}_{ij}^2\widetilde{A}_{ik}^2 - 2\sum_{i,j}\widetilde{A}_{ij}^2\widetilde{A}_{ii}^2 - \sum_{i,j}\widetilde{A}_{ij}^4 + 2\sum_i \widetilde{A}_{ii}^4\\
&= \mathrm{tr}(\widetilde{A}^2\circ\widetilde{A}^2) - 2\mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}^2) - 1_n'[\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}]1_n + 2S_6.
\end{aligned}$$

It is also easy to see that

$$S_3 = \sum_{i,j}\widetilde{A}_{ii}^2\widetilde{A}_{ij}^2 - \sum_i \widetilde{A}_{ii}^4 = \mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}^2) - S_6,$$

$$S_4 = \sum_{i,j} \widetilde{A}_{ij}^4 - \sum_i \widetilde{A}_{ii}^4 = 1_n'[\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}]1_n - S_6,$$

$$S_5 = \sum_{i,j} \widetilde{A}_{ii}\widetilde{A}_{ij}^2\widetilde{A}_{jj} - S_6 = 1_n'[\mathrm{diag}(\widetilde{A})(\widetilde{A}\circ\widetilde{A})\mathrm{diag}(\widetilde{A})]1_n - S_6,$$

$$S_6 = \mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}).$$

Plugging the matrix forms of S_1-S_6 into (8), we obtain

$$Q_n = \mathrm{tr}(\widetilde{A}^4) - 4\mathrm{tr}(\widetilde{A}\circ\widetilde{A}^3) - 2\mathrm{tr}(\widetilde{A}^2\circ\widetilde{A}^2) + 8\mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}^2) - 6\mathrm{tr}(\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}\circ\widetilde{A})$$
$$+ 2\cdot 1_n'[\mathrm{diag}(\widetilde{A})(\widetilde{A}\circ\widetilde{A})\mathrm{diag}(\widetilde{A})]1_n + 1_n'[\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}\circ\widetilde{A}]1_n.$$

This gives the desired expression of Q_n.

Last, we discuss the complexity of computing T_n and Q_n. It involves the following operations:

- Compute the matrix $\widetilde{A} = A - \hat{\eta}\hat{\eta}'$.
- Compute the Hadamard product of finitely many matrices.
- Compute the trace of a matrix.
- Compute the matrix DMD for a matrix M and a diagonal matrix D.
- Compute $1_n'M1_n$ for a matrix M.
- Compute the matrices $\widetilde{A}^k$, for $k = 2, 3, 4$.

Excluding the last operation, the complexity is $O(n^2)$. For the last operation, since we can compute $\widetilde{A}^k$ recursively from $\widetilde{A}^k = \widetilde{A}^{k-1}\widetilde{A}$, it suffices to consider the complexity of computing $B\widetilde{A}$, for an arbitrary $n\times n$ matrix B. Write

$$B\widetilde{A} = BA - B\hat{\eta}(\hat{\eta})'.$$

Consider computing BA. The (i,j)-th entry of BA is $\sum_{\ell: A_{\ell j}\neq 0} B_{i\ell}A_{\ell j}$, where the total number of nonzero $A_{\ell j}$ equals to d_j, the degree of node j. Hence, the complexity of computing the (i,j)-th entry of BA is $O(d_j)$. It follows that the complexity of computing BA is $O(\sum_{i,j=1}^n d_j) = O(n^2\bar{d})$. Consider computing $B\hat{\eta}(\hat{\eta})'$. We first compute the vector $v = B\hat{\eta}$ and then compute $v(\hat{\eta})'$, where the complexity of both steps is $O(n^2)$. Combining the above, the complexity of computing $B\widetilde{A}$ is $O(n^2\bar{d})$. We have seen that this is the dominating step in computing T_n and Q_n, so the complexity of the latter is also $O(n^2\bar{d})$.

APPENDIX D: ESTIMATION OF $\|\theta\|$

We prove Lemma 2.1. First, we show that

$$\|\eta^*\|^2 \begin{cases} = \|\theta\|^2, & \text{under the null,} \\ \asymp \|\theta\|^2, & \text{under the alternative.} \end{cases}$$

Recall that $\eta^* = (1/\sqrt{1_n'\Omega 1_n})\Omega 1_n$. Hence,

$$\|\eta^*\|^2 = (1_n'\Omega^2 1_n)/(1_n'\Omega 1_n). \tag{9}$$

Under the null, $\Omega = \theta\theta'$, and the claim follows by direct calculations. Under the alternative, $\Omega = \sum_{k=1}^K \lambda_k\xi_k\xi_k'$, so

$$1_n'\Omega 1_n = \sum_{k=1}^K \lambda_k(1_n'\xi_k)^2, \qquad 1_n'\Omega^2 1_n = \sum_{k=1}^K \lambda_k^2(1_n'\xi_k)^2.$$

By Lemma E.2, $\lambda_1 \asymp \|\theta\|^2$. By Lemma E.3, $1_n'\xi_1 \asymp \|\theta\|^{-1}\|\theta\|_1$ and $|1_n'\xi_k| = O(\|\theta\|^{-1}\|\theta\|_1)$. It follows that $1_n'\Omega^2 1_n \geq \lambda_1^2(1_n'\xi_1)^2 \geq C\|\theta\|_1^2\|\theta\|^2$ and $1_n'\Omega^2 1_n \leq \lambda_1^2 \sum_{k=1}^K (1_n'\xi_k)^2 \leq C\|\theta\|_1^2\|\theta\|^2$. We conclude that

$$1_n'\Omega^2 1_n \asymp \|\theta\|_1^2\|\theta\|^2. \tag{10}$$

Moreover, $1_n'\Omega 1_n \leq |\lambda_1| \sum_{k=1}^K (1_n'\xi_k)^2 \leq C\|\theta\|_1^2$, and by Lemma E.4, $1_n'\Omega 1_n \geq C\|\theta\|_1^2$. It follows that

$$1_n'\Omega 1_n \asymp \|\theta\|_1^2. \tag{11}$$

Plugging (10)-(11) into (9) gives the claim.

Next, we show $(\|\hat{\eta}\|^2 - 1)/\|\eta^*\|^2 \to 1$ in probability. Since $\|\eta^*\| \asymp \|\theta\| \to \infty$ as $n \to \infty$, it suffices to show $\|\hat{\eta}\|^2/\|\eta^*\|^2 \to 1$ in probability. By definition,

$$\|\hat{\eta}\|^2 = \frac{1_n'A^2 1_n}{1_n'A 1_n}.$$

Compare this with (9), all we need to show is that in probability,

$$\frac{1_n'A1_n}{1_n'\Omega 1_n} \to 1, \qquad \text{and} \qquad \frac{1_n'A^2 1_n}{1_n'\Omega^2 1_n} \to 1. \tag{12}$$

Since the proofs are similar, we only show the second one. By elementary probability, it is sufficient to show that as $n \to \infty$,

$$\frac{\mathbb{E}[1_n'A^2 1_n]}{1_n'\Omega^2 1_n} \to 1, \qquad \frac{\text{Var}(1_n'A^2 1_n)}{(1_n'\Omega^2 1_n)^2} \to 0. \tag{13}$$

We now prove (13). Consider the first claim. Write

$$1_n'A^2 1_n = \sum_{i,j,k} A_{ij}A_{jk} = \sum_{i\neq j} A_{ij}^2 + \sum_{i,j,k(dist)} A_{ij}A_{jk}. \tag{14}$$

It follows that

$$\mathbb{E}[1_n'A^2 1_n] = \sum_{i\neq j} \Omega_{ij} + \sum_{i,j,k(dist)} \Omega_{ij}\Omega_{jk}.$$

Since $\Omega_{ij} \leq \theta_i\theta_j$ under both hypotheses, we have

$$\begin{aligned}
\left|\mathbb{E}[1_n'A^2 1_n] - 1_n'\Omega 1_n - 1_n'\Omega^2 1_n\right| &\leq \Big|\sum_i \Omega_{ii} + \sum_{\substack{(i,j,k) \text{ are} \\ \text{not distinct}}} \Omega_{ij}\Omega_{jk}\Big| \\
&\leq \sum_i \theta_i^2 + C\sum_{i,j} \theta_i^2\theta_j^2 + C\sum_{i,k} \theta_i^3\theta_k \\
&\leq C\|\theta\|^2 + C\|\theta\|^4 + C\|\theta\|_3^3\|\theta\|_1 \\
&\leq C\|\theta\|_3^3\|\theta\|_1,
\end{aligned}$$

where we have used the universal inequality $\|\theta\|^4 \leq \|\theta\|_3^3\|\theta\|_1$. Since $\|\theta\|_3^3 \leq \theta_{\max}^2\|\theta\|_1 = o(\|\theta\|_1)$, the right hand side is $o(\|\theta\|_1^2) = o(1_n'\Omega 1_n)$. So,

$$\mathbb{E}[1_n'A^2 1_n] = 1_n'\Omega^2 1_n + 1_n'\Omega 1_n + o(1_n'\Omega 1_n). \tag{15}$$

Combining this with (10)-(11) gives

$$\left|\frac{\mathbb{E}[1_n'A^2 1_n]}{1_n'\Omega^2 1_n} - 1\right| \lesssim \frac{1_n'\Omega 1_n}{1_n'\Omega^2 1_n} \asymp \frac{1}{\|\theta\|^2},$$

and the claim follows by $\|\theta\| \to \infty$.

Consider the second claim. By (14),

$$\text{Var}(1_n' A^2 1_n) \leq 2\text{Var}\Big(\sum_{i \neq j} A_{ij}^2\Big) + 2\text{Var}\Big(\sum_{i,j,k(dist)} A_{ij}A_{jk}\Big). \tag{16}$$

We re-write $\sum_{i\neq j} A_{ij}^2 = \sum_{i\neq j} A_{ij} = 2\sum_{i<j} A_{ij}$. The variables $\{A_{ij}\}_{1\leq i<j\leq n}$ are mutually independent. It follows that

$$\text{Var}\Big(\sum_{i\neq j} A_{ij}^2\Big) = 4\sum_{i<j}\text{Var}(A_{ij}) \leq C\sum_{i,j}\Omega_{ij} \leq C\|\theta\|_1^2. \tag{17}$$

Moreover, since $A_{ij}A_{jk} = (\Omega_{ij} + W_{ij})(\Omega_{jk} + W_{jk})$, we have

$$\begin{aligned}\sum_{i,j,k(dist)} A_{ij}A_{jk} &= \sum_{i,j,k(dist)} \Omega_{ij}\Omega_{jk} + 2\sum_{i,j,k(dist)} \Omega_{ij}W_{jk} + \sum_{i,j,k(dist)} W_{ij}W_{jk} \\ &\equiv \sum_{i,j,k(dist)} \Omega_{ij}\Omega_{jk} + X_1 + X_2.\end{aligned}$$

By elementary probability,

$$\text{Var}\Big(\sum_{i,j,k(dist)} A_{ij}A_{jk}\Big) \leq 2\text{Var}(X_1) + 2\text{Var}(X_2).$$

To compute the variance of X_1, we note that

$$X_1 = 4\sum_{j<k}\beta_{jk}W_{jk}, \qquad \beta_{jk} = \sum_{i\notin\{j,k\}}\Omega_{ij}.$$

The variables $\{W_{jk}\}_{1\leq j<k\neq n}$ are mutually independent, and $|\beta_{jk}| \leq C\sum_i \theta_i\theta_j \leq C\|\theta\|_1\theta_j$. It follows that

$$\text{Var}(X_1) \leq C\sum_{j,k}(\|\theta\|_1\theta_j)^2(\theta_j\theta_k) \leq C\|\theta\|_1^3\|\theta\|_3^3.$$

To compute the variance of X_2, we note that

$$\text{Var}(X_2) = \sum_{i,j,k(dist)}\sum_{i',j',k'(dist)} \mathbb{E}[W_{ij}W_{jk}W_{i'j'}W_{j'k'}].$$

The summand is nonzero only when the two variables $\{W_{i'j'}, W_{j'k'}\}$ are the same as the two variables $\{W_{ij}, W_{jk}\}$. This can only happen if $(i,j,k) = (i',j',k')$ or $(i,j,k) = (k',j',i')$, where in either case the summand equals to $\mathbb{E}[W_{ij}^2W_{jk}^2]$. It follows that

$$\text{Var}(X_2) = \sum_{i,j,k(dist)} 2\mathbb{E}[W_{ij}^2W_{jk}^2] \leq C\sum_{i,j,k}\theta_i\theta_j^2\theta_k \leq C\|\theta\|^2\|\theta\|_1^2.$$

Combining the above gives

$$\text{Var}\Big(\sum_{i,j,k(dist)} A_{ij}A_{jk}\Big) \leq C\|\theta\|_1^3\|\theta\|_3^3 + C\|\theta\|^2\|\theta\|_1^2 \leq C\|\theta\|_1^3\|\theta\|_3^3, \tag{18}$$

where we have used the fact that $\|\theta\|_1\|\theta\|_3^3 \geq \|\theta\|^4$ (Cauchy-Schwarz inequality) and $\|\theta\| \to \infty$. Plugging (17)-(18) into (16) gives

$$\text{Var}(1_n' A^2 1_n) \leq C\|\theta\|_1^3\|\theta\|_3^3. \tag{19}$$

Comparing this with (10) and using $\|\theta\|_3^3 \leq \theta_{\max}^2\|\theta\|_1$, we obtain

$$\frac{\text{Var}(1_n' A^2 1_n)}{(1_n'\Omega^2 1_n)^2} \leq \frac{C\|\theta\|_1^3\|\theta\|_3^3}{\|\theta\|_1^4\|\theta\|^4} \leq \frac{C\theta_{\max}^2}{\|\theta\|^4},$$

and the claim follows by $\|\theta\| \to \infty$.

APPENDIX E: SPECTRAL ANALYSIS FOR Ω AND $\widetilde{\Omega}$

We state and prove some useful results about eigenvalues and eigenvectors of Ω and $\widetilde{\Omega}$. In Section E.4, we prove Lemma 2.2 and 2.3 of the main file.

For $1 \le k \le K$, let λ_k be the k-th largest (in absolute value) eigenvalue of Ω and let $\xi_k \in \mathbb{R}^n$ be the corresponding unit-norm eigenvector. We write

$$\Xi = [\xi_1, \xi_2, \dots, \xi_K] = [u_1, u_2, \dots, u_n]',$$

so that u_i is the i-th row of Ξ. Recall that G is the $K \times K$ matrix $\|\theta\|^{-2}(\Pi'\Theta^2\Pi)$.

E.1. Spectral analysis of Ω. The following lemma relates λ_k and ξ_k to the eigenvalues and eigenvectors of the $K \times K$ matrix $G^{\frac{1}{2}}PG^{\frac{1}{2}}$.

LEMMA E.1. *Consider the DCMM model. Let d_k be the k-th largest (in absolute value) eigenvalue of $G^{\frac{1}{2}}PG^{\frac{1}{2}}$ and let $\beta_k \in \mathbb{R}^K$ be the associated eigenvector, $1 \le k \le K$. Then under the null,*

$$\lambda_1 = \|\theta\|^2, \qquad \xi_1 = \pm\theta/\|\theta\|.$$

Under the alternative, for $1 \le k \le K$,

$$\lambda_k = d_k\|\theta\|^2, \qquad \xi_k = \|\theta\|^{-1}[\theta \circ (\Pi G^{-\frac{1}{2}}\beta_k)].$$

Under the alternative hypothesis, we further have the following lemma:

LEMMA E.2. *Under the DCMM model, as $n \to \infty$, suppose (2.2) holds. As $n \to \infty$, under the alternative hypothesis,*

$$\lambda_1 \asymp \|\theta\|^2, \qquad \|u_i\| \le C\|\theta\|^{-1}\theta_i, \quad \textit{for all } 1 \le i \le n.$$

The quantities $(1_n'\xi_k)$ play key roles in the analysis of the Signed Polygon tests. By Lemma E.1,

$$\xi_1 = (\|\theta\|)^{-1}\Theta\Pi G^{-1/2}\beta_1,$$

where β_1 is the first eigenvector of $G^{1/2}PG^{1/2}$, corresponding to the largest eigenvalue of $G^{1/2}PG^{1/2}$. It is seen $G^{-1/2}\beta_1$ is the eigenvector of the matrix PG associated with the largest eigenvalue of GP, which is the same as the largest eigenvalue of $G^{1/2}PG^{1/2}$. Since PG is a non-negative matrix, by Perron's theorem, we can assume all entries of $G^{-1/2}\beta_1$ are non-negative. As a result, all entries of ξ_1 are non-negative, and

$$1_n'\xi_1 > 0.$$

The following lemma is proved in Section E.3.

LEMMA E.3. *Under the DCMM model, as $n \to \infty$, suppose (2.2) holds. As $n \to \infty$,*

$$\max_{1\le k\le K} |1_n'\xi_k| \le C\|\theta\|^{-1}\|\theta\|_1, \qquad 1_n'\xi_1 \ge C\|\theta\|^{-1}\|\theta\|_1.$$

and so for any $2 \le k \le K$,

$$|1_n'\xi_k| \le C|1_n'\xi_1|$$

We also have a lower bound for $1_n'\Omega 1_n$. The following lemma is proved in Section E.3.

LEMMA E.4. *Under the DCMM model, as $n \to \infty$, suppose (2.2) holds. As $n \to \infty$, both under the null hypothesis and the alternative hypothesis,*

$$1_n'\Omega 1_n \ge C\|\theta\|_1^2.$$

E.2. Spectral analysis of $\tilde{\Omega}$. Recall that

$$\widetilde{\Omega} = \Omega - (\eta^*)(\eta^*)', \qquad \text{where } \eta^* = (1/\sqrt{1_n'\Omega 1_n})\Omega 1_n,$$

and $\lambda_1, \ldots, \lambda_K$ are the K nonzero eigenvalues of Ω, arranged in the descending order in magnitude, and $\xi_1, \ldots, \xi_K$ are the corresponding unit-norm eigenvectors of Ω The following lemma is proved in Section E.3.

LEMMA E.5. *Under the DCMM model, as $n \to \infty$, suppose (2.2) holds. Then,*

$$|\lambda_2| \le \|\widetilde{\Omega}\| \le C|\lambda_2|.$$

Moreover, for any fixed integer $m \ge 1$,

$$|(\widetilde{\Omega}^m)_{ij}| \le C|\lambda_2|^m \cdot \|\theta\|^{-2}\theta_i\theta_j, \quad \textit{for all } 1 \le i, j \le n.$$

Recall that $d_1, \ldots, d_K$ are the nonzero eigenvalues of $G^{\frac{1}{2}}PG^{\frac{1}{2}}$. Introduce

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_K), \qquad \widetilde{D} = \mathrm{diag}(d_2, d_3, \ldots, d_K),$$

and

$$h = \Big(\frac{1_n'\xi_2}{1_n'\xi_1}, \frac{1_n'\xi_3}{1_n'\xi_1}, \ldots, \frac{1_n'\xi_K}{1_n'\xi_1}\Big)', \qquad u_0 = \sum_{k=2}^{K} \frac{d_k(1_n'\xi_k)^2}{d_1(1_n'\xi_1)^2}.$$

By Lemma E.3, $1_n'\xi_1 > 0$, so h and u_0 are both well-defined. Write $\Xi = [\xi_1, \xi_2, \ldots, \xi_K]$. The following lemma gives an alternative expression of $\widetilde{\Omega}$.

LEMMA E.6. *Under the DCMM model,*

$$\widetilde{\Omega} = \|\theta\|^2 \cdot \Xi M \Xi',$$

where M is a $K \times K$ matrix satisfying

$$M = \begin{bmatrix} (1+u_0)^{-1}h'\widetilde{D}h & -(1+u_0)^{-1}h'\widetilde{D} \\ -(1+u_0)^{-1}\widetilde{D}h & \widetilde{D} - (d_1(1+u_0))^{-1}\widetilde{D}hh'\widetilde{D} \end{bmatrix}.$$

If additionally $|\lambda_2|/\lambda_1 \to 0$, then for the matrix $\widetilde{M} \in \mathbb{R}^{K,K}$,

$$\widetilde{M} = \|\theta\|^2 \cdot \begin{bmatrix} h'\widetilde{D}h & -h'\widetilde{D} \\ -\widetilde{D}h & \widetilde{D} \end{bmatrix},$$

we have

$$|M_{ij} - \widetilde{M}_{ij}| \le C\lambda_2^2/\lambda_1, \qquad \textit{for all } 1 \le i, j \le K.$$

We now study $\mathrm{tr}(\widetilde{\Omega}^3)$ and $\mathrm{tr}(\widetilde{\Omega}^4)$. They are related to the power of the SgnT test and SgnQ test, respectively. We discuss the two cases $|\lambda_2|/\lambda_1 \to 0$ and $|\lambda_2|/\lambda_1 \ge c_0$ separately. Consider the case of $|\lambda_2|/\lambda_1 = o(1)$. Since $\widetilde{\Omega} = \Xi M \Xi'$, where $\Xi'\Xi = I_K$, we have

$$\mathrm{tr}(\widetilde{\Omega}^3) = \mathrm{tr}(M^3), \qquad \text{and} \qquad \mathrm{tr}(\widetilde{\Omega}^4) = \mathrm{tr}(M^4).$$

The following lemma is proved in Section E.3.

LEMMA E.7. *Consider the DCMM model, where (2.2) holds. As $n\to\infty$, if $|\lambda_2|/\lambda_1\to 0$, then*

$$|\mathrm{tr}(\widetilde{\Omega}^3)-\mathrm{tr}(\widetilde{M}^3)|\leq o(|\lambda_2|^3),\qquad |\mathrm{tr}(\widetilde{\Omega}^4)-\mathrm{tr}(\widetilde{M}^4)|\leq o(|\lambda_2|^3), \tag{20}$$

Moreover,

$$\mathrm{tr}(\widetilde{M}^3)=\mathrm{tr}(\widetilde{D}^3)+3h'\widetilde{D}^3h+3(h'\widetilde{D}h)(h'\widetilde{D}^2h)+(h'\widetilde{D}h)^3,$$

and

$$\begin{aligned}\mathrm{tr}(\widetilde{M}^4)&=\mathrm{tr}(\widetilde{D}^4)+(h'\widetilde{D}h)^4+4(h'\widetilde{D}^2h)^2+4(h'\widetilde{D}h)^2(h'\widetilde{D}^2h)+4h'\widetilde{D}^4h+4(h'\widetilde{D}h)(h'\widetilde{D}^3h)\\&\geq \mathrm{tr}(\widetilde{D}^4)+(h'\widetilde{D}h)^4+2[(h'\widetilde{D}^2h)^2+(h'\widetilde{D}h)^2(h'\widetilde{D}^2h)+h'\widetilde{D}^4h]\\&\geq \mathrm{tr}(\widetilde{D}^4).\end{aligned}$$

- *In the special case where $\lambda_2,\lambda_3,\ldots,\lambda_K$ have the same signs,*

$$|\mathrm{tr}(\widetilde{M}^3)|\geq |\sum_{k=2}^K\lambda_k^3|=\sum_{k=2}^K|\lambda_k|^3,$$

and so

$$|\mathrm{tr}(\widetilde{\Omega}^3)|\geq \sum_{k=2}^K|\lambda_k|^3+o(|\lambda_2|^3).$$

- *In the special case where $K=2$, the vector h is a scalar, and*

$$\mathrm{tr}(\widetilde{M}^3)=(1+h^2)^3\lambda_2^3,\qquad \mathrm{tr}(\widetilde{M}^4)=(1+h^2)^4\lambda_2^4,$$

and so

$$\mathrm{tr}(\widetilde{\Omega}^3)=[(1+h^2)^3+o(1)]\lambda_2^3,\qquad \mathrm{tr}(\widetilde{\Omega}^4)=[(1+h^2)^4+o(1)]\lambda_2^4.$$

We now consider the case $|\lambda_2/\lambda_1|\geq c_0$. In this case, $\widetilde{M}$ is not a good proxy for M any more, so we can not derive a simple formula for $\mathrm{tr}(\widetilde{\Omega}^3)$ or $\mathrm{tr}(\widetilde{\Omega}^4)$ as above. However, for $\mathrm{tr}(\widetilde{\Omega}^4)$, since

$$\mathrm{tr}(\widetilde{\Omega}^4)\geq \|\widetilde{\Omega}\|^4,$$

by Lemma E.5, we immediately have

$$\mathrm{tr}(\widetilde{\Omega}^4)\geq C\lambda_2^4\geq C(\sum_{k=2}^K\lambda_k^4)/(K-1)\geq C\sum_{k=2}^K\lambda_k^4. \tag{21}$$

E.3. Proof of Lemmas E.1-E.7.

E.3.1. *Proof of Lemma E.1.* The proof for the null case is straightforward, so we only prove the lemma for the alternative case. Consider the spectral decomposition

$$G^{1/2}PG^{1/2}=BDB'.$$

where

$$D=\mathrm{diag}(d_1,\ldots,d_K)\qquad\text{and}\qquad B=[\beta_1,\ldots,\beta_K].$$

Combining this with $\Omega = \Theta\Pi P\Pi'\Theta$ gives

$$\begin{aligned}\Omega &= \Theta\Pi G^{-\frac{1}{2}}(G^{\frac{1}{2}}PG^{\frac{1}{2}})G^{-\frac{1}{2}}\Pi'\Theta \\ &= \Theta\Pi G^{-\frac{1}{2}}(BDB')G^{-\frac{1}{2}}\Pi'\Theta \\ &= (\|\theta\|^{-1}\Theta\Pi G^{-\frac{1}{2}}B)(\|\theta\|^2 D)(\|\theta\|^{-1}\Theta\Pi G^{-\frac{1}{2}}B)' \\ &= H(\|\theta\|^2 D)H',\end{aligned}$$

where

$$H = \|\theta\|^{-1}\Theta\Pi G^{-\frac{1}{2}}B.$$

Recalling that $G = (\|\theta\|^2)^{-1}\cdot\Pi'\Theta^2\Pi$, it is seen

$$H'H = \|\theta\|^{-2}B'G^{-\frac{1}{2}}(\Pi'\Theta^2\Pi)G^{-\frac{1}{2}}B = B'B = I_K, \tag{22}$$

Therefore,

$$\Omega = H(\|\theta\|^2 D)H'$$

is the spectral decomposition of Ω. Since $(\widetilde{D}_k, \xi_k)$ are the k-th eigenvalue of Ω and unit-norm eigenvector respectively, we have

$$\xi_k = \pm 1 \cdot \text{the } k\text{-th column of H} = \pm(\|\theta\|)^{-1}\Theta\Pi G^{-1/2}\beta_k.$$

This proves the claim. □

E.3.2. *Proof of Lemma E.2.* Consider the first claim. By Lemma E.1, $\lambda_1 = d_1\|\theta\|^2$, where d_1 is the maximum eigenvalue of $G^{\frac{1}{2}}PG^{\frac{1}{2}}$. It suffices to show that $d_1 \asymp 1$. Since all entries of P are upper bounded by constants, we have

$$\|P\| \leq C.$$

Additionally, since G is a nonnegative symmetric matrix,

$$\|G\| \leq \|G\|_{\max} = \max_{1\leq k\leq K}\sum_{\ell=1}^K G(k,\ell) = \|\theta\|^{-2}\max_{1\leq k\leq K}\sum_{\ell=1}^K\sum_{i=1}^n \pi_i(k)\pi_i(\ell)\theta_i^2 \leq 1. \tag{23}$$

It follows that

$$d_1 \leq \|G\|\|P\| \leq C. \tag{24}$$

At the same time, for any unit-norm non-negative vector $x \in \mathbb{R}^K$, since all entries of P are non-negative and all diagonal entries of P are 1,

$$x'Px \geq x'x = 1.$$

It follows that

$$d_1 = \|G^{\frac{1}{2}}PG^{\frac{1}{2}}\| \geq \frac{(G^{-\frac{1}{2}}x)'(G^{\frac{1}{2}}PG^{\frac{1}{2}})(G^{-\frac{1}{2}}x)}{\|(G^{-\frac{1}{2}}x)\|^2} = \frac{x'Px}{x'G^{-1}x} \geq \frac{1}{\|G^{-1}\|}.$$

Combining it with the assumption (2.2) gives

$$d_1 \geq C. \tag{25}$$

where we note C denotes a generic constant which may vary from occurrence to occurrence. Combining (24)-(25) gives the claim.

Consider the second claim. Let $B = [\beta_1, \beta_2, \ldots, \beta_K]$ and $D = \text{diag}(d_1, d_2, \ldots, d_K)$ as in the proof of Lemma E.1, where we note B is orthonormal. By Lemma E.1 and definitions,

$$u_i' = \|\theta\|^{-1}\theta_i \pi_i' G^{-\frac{1}{2}} B.$$

It follows that

$$\|u_i\| \leq \|\theta\|^{-1}\theta_i \cdot \|\pi_i\| \|G^{-\frac{1}{2}}\| \|B\| \leq (\|\theta\|)^{-1}\theta_i \|G^{-1/2}\|,$$

where we have used $\|B\| = 1$ and $\|\pi_i\| = [\sum_{k=1}^K \pi_i(k)^2]^{1/2} \leq 1$. Finally, by the assumption (2.2), $\|G^{-1}\| \leq C$ and so $\|G^{-1/2}\| \leq C$. Combining these gives the claim. □

E.3.3. *Proof of Lemma E.3.* It is sufficient to show the first two claims. Consider the first claim. By Lemma E.2, for all $1 \leq k \leq K$ and $1 \leq i \leq n$,

$$|\xi_k(i)| \leq C\|\theta\|^{-1}\theta_i.$$

It follows that

$$|1_n'\xi_k| \leq C\sum_{i=1}^n \|\theta\|^{-1}\theta_i \leq C\|\theta\|^{-1}\|\theta\|_1, \qquad \text{for all } 1 \leq k \leq K, \tag{26}$$

and the claim follows.

Consider the second claim. By Lemma E.1,

$$\xi_1 = \|\theta\|^{-1}\Theta\Pi(G^{-\frac{1}{2}}\beta_1), \tag{27}$$

where β_1 is the (unit-norm) eigenvector of $G^{\frac{1}{2}}PG^{\frac{1}{2}}$ associated with λ_1, which is the largest eigenvalue of $G^{1/2}PG^{1/2}$. By basic algebra, λ_1 is also the largest eigenvalue of the matrix PG, with $G^{-1/2}\beta_1$ being the corresponding eigenvector. Since PG is a nonnegative matrix, $G^{-\frac{1}{2}}\beta_1$ is a nonnegative vector (e.g., [2, Theorem 8.3.1]). Denote for short by

$$h = G^{-1/2}\beta_1.$$

It follows from (27) that

$$1_n'\xi_1 = (\|\theta\|)^{-1} \cdot 1_n'\Theta\Pi h = \|\theta\|^{-1} \cdot \sum_{k=1}^K \Big(\sum_{i=1}^n \pi_i(k)\theta_i\Big) h_k. \tag{28}$$

We note that $\sum_{k=1}^K \big(\sum_{i=1}^n \pi_i(k)\theta_i\big) = \|\theta\|_1$. Combining it with the assumption (2.2) yields

$$\min_{1\leq k\leq K}\Big\{\sum_{i=1}^n \pi_i(k)\theta_i\Big\} \geq C\|\theta\|_1.$$

Inserting this into (28) gives

$$1_n'\xi_1 \geq C(\|\theta\|)^{-1}\|\theta\|_1 \cdot \|h\|_1. \tag{29}$$

We claim that $\|h\| \geq 1$. Otherwise, if $\|h\| < 1$, then every entry of h is no greater than 1 in magnitude, and so

$$\|h\|_1 \geq \|h\|^2 = \|G^{-1}\beta_1\|^2.$$

Since $\|G^{-1}\| = \|G\|^{-1} \geq 1$ (see (23)) and $\|\beta_1\| = 1$,

$$\|G^{-\frac{1}{2}}\beta_1\| \geq 1.$$

and so it follows $\|h\| \geq 1$. The contradiction show that $\|h\| \geq 1$. The claim follows by combining this with (29). □

E.3.4. *Proof of Lemma E.4.* For $1 \leq k \leq K$, let

$$c = (\|\theta\|_1)^{-1}\Pi'\Theta 1_n = (\|\theta\|_1)^{-1}(1_n'\Theta\Pi)'.$$

Since $\Omega = \Theta\Pi P\Pi'\Theta$ and all entries of P are non-negative,

$$1_n'\Omega 1_n = \|\theta\|_1^2(c'Pc) \geq \|\theta\|^2\Big(\sum_{k=1}^{K} c_k^2\Big). \tag{30}$$

Note that, first, $c_k \geq 0$, and second, $\|\theta\|_1 \sum_{k=1}^{K} c_k = 1_n'\Pi\Theta 1_n = 1_n'\Theta 1_n$, where the last term is $\|\theta\|_1$, and so

$$\sum_{k=1}^{K} c_k = 1.$$

Together with the Cauchy-Schwartz inequality, we have

$$\sum_{k=1}^{K} c_k^2 \geq (\sum_{k=1}^{K} c_k)^2/K = 1/K.$$

Combining this with (30) gives the claim. $\square$

E.3.5. *Proof of Lemma E.5.* Consider the first claim. We first derive a lower bound for $\|\widetilde{\Omega}\|$. By Lemma E.6,

$$\|\widetilde{\Omega}\| = \|\theta\|^2 \cdot \|M\|, \tag{31}$$

where with the same notations as in the proof of Lemma E.6, $M = D - (1+u_0)^{-1}vv'$. Let M_0 be the top left 2×2 block of M. Let $D_0 = \text{diag}(d_1, d_2)$, and let v_0 be the sub-vector of v in (36) restricted to the first two coordinates. By (36),

$$M_0 = D_0 - (1+u_0)^{-1}v_0v_0' = D_0^{\frac{1}{2}}\Big(I_2 - (1+u_0)^{-1}D_0^{-1/2}v_0v_0'D_0^{-\frac{1}{2}}\Big)D_0^{\frac{1}{2}},$$

and so by $\|D_0^{-1/2}\| = |d_2|^{-1/2}$ we have

$$\|\Big(I_2 - (1+u_0)^{-1}D_0^{-1/2}v_0v_0'D_0^{-\frac{1}{2}}\Big)\| \leq \|D_0^{-1/2}M_0D_0^{-1/2}\| \leq |d_2|^{-1} \cdot \|M_0\|. \tag{32}$$

At the same time, since $(1+u_0)^{-1}D_0^{-1/2}v_0v_0'D_0^{-1/2}$ is a rank-1 matrix, there is an orthonormal matrix and a number b such that

$$Q(1+u_0)^{-1}D_0^{-1/2}v_0v_0'D_0^{-1/2}Q' = \text{diag}(b, 0).$$

It follows

$$\|\Big(I_2 - (1+u_0)^{-1}D_0^{-1/2}v_0v_0'D_0^{-\frac{1}{2}}\Big)\| = \|I_2 - \text{diag}(b,0)\| = \max\{|1-b|, 1\} \geq 1.$$

Inserting this into (32) gives

$$\|M_0\| \geq |d_2|,$$

Note that $\|M\| \geq \|M_0\|$. Combining this with (31) gives

$$\|\widetilde{\Omega}\| \geq |d_2|\|\theta\|^2.$$

Next, we derive an upper bound for $\|\widetilde{\Omega}\|$. By Lemma E.3,

$$\max_{1\leq k\leq K}|1_n'\xi_k| \leq C\|\theta\|^{-1}\|\theta\|_1, \qquad 1_n'\xi_1 \geq C\|\theta\|^{-1}\|\theta\|_1. \tag{33}$$

By (33), all the entries of M are upper bounded by $C|\lambda_2|$, which implies $\|M\| \leq C|d_2|$. Plugging it into (31) gives

$$\|\widetilde{\Omega}\| \leq \frac{C}{|1+u_0|}|d_2|\|\theta\|^2, \tag{34}$$

and all remains to show is

$$1 + u_0 \geq C > 0.$$

Now, recalling that $\Omega = \sum_{k=1}^K \lambda_k \xi_k \xi_k'$ and $\lambda_k = d_k\|\theta\|^2$, by definitions,

$$d_1(1_n'\xi_1)^2(1+u_0) = \sum_{k=1}^K d_k(1_n'\xi_k)^2 = \|\theta\|^{-2}1_n'\Omega 1_n.$$

By Lemma E.4 which gives $1_n'\Omega 1_n \geq C\|\theta\|_1^2$. It follows that

$$1 + u_0 \geq \frac{\|\theta\|^{-2}1_n'\Omega 1_n}{d_1(1_n'\xi_1)^2} \geq C\frac{\|\theta\|^{-2}\cdot\|\theta\|_1^2}{\|\theta\|^{-2}\cdot\|\theta\|_1^2} \geq C,$$

where in the second inequality we have used (33) and $d_1 = (\|\theta\|)^{-2}\cdot\lambda_1 \leq 1$ (see Lemma E.2). Inserting this into (34) gives the claim.

Consider the second claim. By Lemma E.6,

$$\widetilde{\Omega} = \Xi M \Xi',$$

where Ξ and M are the same there. Write

$$\Xi = [\xi_1, \xi_2, \ldots, \xi_K] = [u_1, u_2, \ldots, u_n]'.$$

Note that $\widetilde{\Omega}$ and M have the same spectral norm. It follows that

$$\widetilde{\Omega}^m = \Xi M^m \Xi',$$

and

$$|(\widetilde{\Omega}^m)_{ij}| = |u_i' M^m u_j| \leq \|u_i\|\|M\|^m\|u_j\|.$$

By Lemma E.2, $\|u_i\|\|u_j\| \leq C\|\theta\|^{-2}\theta_i\theta_j$, and by the first part of the current lemma,

$$\|M\| = \|\tilde{\Omega}\| \leq C|d_2|\|\theta\|^2.$$

It follows that

$$|(\widetilde{\Omega}^m)_{ij}| \leq C|d_2|^m\|\theta\|^{2m-2}\theta_i\theta_j.$$

This proves the claim. □

E.3.6. *Proof of Lemma E.6.* Consider the first claim. By definitions,

$$\widetilde{\Omega} = \Omega - (\eta^*)(\eta^*)', \qquad \text{where } \eta^* = \frac{1}{\sqrt{1_n'\Omega 1_n}}\Omega 1_n. \tag{35}$$

Recalling $\widetilde{D}_k = d_k\|\theta\|^2$ and $\Xi = [\xi_1, \xi_2, \ldots, \xi_K]$, we have

$$\Omega = \sum_{k=1}^K \widetilde{D}_k \xi_k \xi_k' = \|\theta\|^2 \cdot \Xi D \Xi'.$$

It follows that

$$1_n'\Omega 1_n = \|\theta\|^2 \sum_{k=1}^K d_k(1_n'\xi_k)^2,$$

and

$$\eta^* = \frac{\|\theta\|}{\sqrt{\sum_{s=1}^K d_s(1_n'\xi_s)^2}} \sum_{k=1}^K d_k(1_n'\xi_k)\xi_k = \frac{\|\theta\|}{\sqrt{(1+u_0)}}\Big[\sqrt{d_1}\,\xi_1 + \sum_{k=2}^K \frac{d_k(1_n'\xi_k)}{\sqrt{d_1}(1_n'\xi_1)}\xi_k\Big],$$

where the vector in the big bracket on the right is Ξv, if we let

$$v = (\sqrt{d_1}, \frac{d_2(1_n'\xi_2)}{\sqrt{d_1}(1_n'\xi_1)}, \ldots, \frac{d_K(1_n'\xi_K)}{\sqrt{d_1}(1_n'\xi_1)})'.$$

Combining these gives

$$\widetilde{\Omega} = \|\theta\|^2 \Xi D \Xi' - \frac{\|\theta\|^2}{1+u_0}\Xi v v'\Xi.$$

Plugging it into (35) gives

$$\widetilde{\Omega} = \|\theta\|^2 \Xi D \Xi' - \frac{\|\theta\|^2}{1+u_0}\Xi v v'\Xi = \|\theta\|^2 \Xi (D - (1+u_0)^{-1} v v')\Xi'. \tag{36}$$

By definitions,

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_K), \qquad \text{and} \qquad v = d_1^{-1/2}\cdot(d_1, h'\widetilde{D})'.$$

It follows

$$D - (1+u_0)^{-1}vv' = \begin{bmatrix} (1+u_0)^{-1}d_1 u_0 & -(1+u_0)^{-1}h'\widetilde{D} \\ -(1+u_0)^{-1}\widetilde{D}h & \widetilde{D} - (d_1(1+u_0))^{-1}\widetilde{D}hh'\widetilde{D} \end{bmatrix},$$

where we note that

$$d_1 u_0 = \sum_{s=2}^K d_s \frac{(1_n'\xi_s)^2}{(1_n'\xi_1)^2} = h'\widetilde{D}h,$$

Combining these gives the claim.

Consider the second claim. By definitions,

$$M - \widetilde{M} = \|\theta\|^2 \cdot \begin{bmatrix} [(1+u_0)^{-1} - 1]d_1 u_0 & (1 - (1+u_0)^{-1})h'\widetilde{D} \\ (1-(1+u_0)^{-1})\widetilde{D}h & -(d_1(1+u_0))^{-1}\widetilde{D}hh'\widetilde{D} \end{bmatrix}.$$

Note that

$$|1 - (1+u_0)^{-1}| \le C|u_0| \le C|\widetilde{D}_2|/\widetilde{D}_1,$$

and that by Lemma E.3,

$$|(1_n'\xi_k)| \le C 1_n'\xi_1,$$

and so each entry of $\widetilde{D}h$ does not exceed $C|d_2|$. It follows that for all $2 \le i, j \le K$,

$$|M_{1i} - \widetilde{M}_{1i}| \le C\|\theta\|^2(|\widetilde{D}_2|/\widetilde{D}_1)d_2^2 \le C\widetilde{D}_2^2/\widetilde{D}_1,$$

and

$$|M_{ij} - \widetilde{M}_{ij}| \le C\|\theta\|^2 d_1^{-1} d_2^2 \le C\widetilde{D}_2^2/\widetilde{D}_1.$$

Finally,

$$d_1 u_0^2 = d_1^{-1}(\sum_{s=2} d_2 \frac{(1_n'\xi_s)^2}{(1_n'\xi_1)^2})^2 \leq C d_2^2/d_1,$$

so

$$|M_{11} - \widetilde{M}_{11}| \leq C\|\theta\|^2 d_2^2/d_1 \leq C\widetilde{D}_2^2/\widetilde{D}_1.$$

Combining these gives the claim. □

E.3.7. *Proof of Lemma E.7.* It is sufficient to show (20). In fact, once (20) is proved, other claims follow by direct calculations, except for the first inequality regarding $\mathrm{tr}(\widetilde{\Omega}^4)$, we have used

$$|(h'\widetilde{D}h)(h'\widetilde{D}^3h)| \leq |h'\widetilde{D}h|\sqrt{(h'\widetilde{D}^2h)(h'\widetilde{D}^4h)} \leq \frac{1}{2}\Big[(h'\widetilde{D}h)^2(h'\widetilde{D}^2h) + h'\widetilde{D}^4h\Big].$$

We now show (20). Since $\mathrm{tr}(\widetilde{\Omega}^m) = \mathrm{tr}(\widetilde{M}^m)$, for $m = 3, 4$, it is sufficient to show

$$|\mathrm{tr}(M^3) - \mathrm{tr}(\widetilde{M}^3)| \leq C\lambda_2^4/\lambda_1), \qquad |\mathrm{tr}(M^4) - \mathrm{tr}(\widetilde{M}^4)| \leq C|\lambda_2|^5/\lambda_1. \tag{37}$$

Since the proofs are similar, we only show the first one. By basic algebra,

$$\mathrm{tr}(M^3 - \widetilde{M}^3) = \mathrm{tr}((M - \widetilde{M})^3) + 3\mathrm{tr}(\widetilde{M}(M - \widetilde{M})^2) + 3\mathrm{tr}(\widetilde{M}^2(M - \widetilde{M})).$$

By Lemma E.6, for all $1 \leq i, j \leq K$,

$$|M_{ij} - \widetilde{M}_{ij}| \leq C\lambda_2^2/\lambda_1.$$

Also, by Lemma E.3, all entries of h are bounded, so for all $1 \leq i, j \leq K$,

$$|\widetilde{M}_{ij}| \leq |\lambda_2|.$$

It follows

$$|\mathrm{tr}((M - \widetilde{M})^3| \leq C(\lambda_2^2/\lambda_1)^3,$$

$$|\mathrm{tr}(\widetilde{M}(M - \widetilde{M})^2)| \leq C|\lambda_2|(\lambda^2/\lambda_1)^2 \leq C|\lambda_2|^5/\lambda_1^2,$$

and

$$|\mathrm{tr}(\widetilde{M}^2(M - \widetilde{M})| \leq C\lambda_2^2(\lambda^2/\lambda_1) \leq C\lambda_2^4/\lambda_1.$$

where we note that $\lambda_2/\lambda_1 = o(1)$. Combining these gives the claim.

E.4. Proof of Lemmas 2.2 and 2.3. Lemma 2.2 follows directly from Lemma E.7 of this appendix. Consider Lemma 2.3. The second bullet point is a direct result of (21), and the other two bullet points follow directly from Lemma E.7 of this appendix.

APPENDIX F: LOWER BOUNDS, REGION OF IMPOSSIBILITY

We study the Region of Impossibility by considering a DCMM with random mixed memberships. First, in Section F.1, we establish the equivalence between regularity conditions for a DCMM with non-random mixed memberships and those for a DCMM with random mixed memberships. Next, we prove Lemma 3.1, which is key to the construction of inseparable hypothesis pairs. Last, we prove Theorems 3.1-3.5 in the main article.

F.1. Equivalence of regularity conditions. Let $\mu_1, \mu_2, \dots, \mu_K$ be the eigenvalues of P, arranged in the descending order in magnitude. Recall that $\lambda_1, \lambda_2, \dots, \lambda_K$ are the eigenvalues of Ω. The following lemma is proved in Section F.5.

LEMMA F.1 (Equivalent definition of Region of Impossibility). *Consider the DCMM model (1.1)-(1.4), where the alternative is true and the condition (2.2) holds. Suppose $\theta_{\max} \to 0$ and $\|\theta\| \to \infty$ as $n \to \infty$. Then, as $n \to \infty$,*

$$\mu_1 \asymp 1, \qquad \frac{|\mu_2|}{\mu_1} \asymp \frac{|\lambda_2|}{\lambda_1}, \qquad \max_{1 \le i,j \le K} |P_{ij} - 1| \le C(|\lambda_2|/\lambda_1).$$

As a result, $|\lambda_2|/\sqrt{\lambda_1} \to 0$ if and only if $\|\theta\| \cdot |\mu_2(P)| \to 0$.

We now consider DCMM with random mixed memberships: Given (Θ, P) and a distribution F over V (the standard simplex in $\mathbb{R}^K$), let

$$\Omega = \Theta \Pi P \Pi' \Theta, \qquad \Pi = [\pi_1, \pi_2, \dots, \pi_n]', \qquad \pi_i \overset{iid}{\sim} F. \tag{38}$$

We notice that the conclusion of Lemma F.1 holds provided that the regularity condition (2.2) is satisfied. The next lemma shows that (2.2) holds with high probability. It is proved in Section F.5.

LEMMA F.2 (Equivalence of regularity conditions). *Consider the model (38). Let $h = \mathbb{E}[\pi_i]$ and $\Sigma = \mathbb{E}[\pi_i \pi_i']$. Suppose $\|P\| \le C$, $\min_{1 \le k \le K}\{h_k\} \ge C$ and $\|\Sigma^{-1}\| \le C$. Suppose $\theta_{\max} \to 0$, $\|\theta\| \to \infty$, and $(\|\theta\|^2/\|\theta\|_1)\sqrt{\log(\|\theta\|_1)} \to 0$, as $n \to \infty$. Then, as $n \to \infty$, with probability $1 - o(1)$, the condition (2.2) is satisfied, i.e.,*

$$\frac{\max_{1 \le k \le K}\{\sum_{i=1}^n \theta_i \pi_i(k)\}}{\min_{1 \le k \le K}\{\sum_{i=1}^n \theta_i \pi_i(k)\}} \le C_0, \qquad \|G^{-1}\| \le C_0,$$

for a constant $C_0 > 0$ and $G = \|\theta\|^{-2}(\Pi' \Theta^2 \Pi)$.

F.2. Proof of Lemma 3.1. Let $M = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_K)$. It is seen $\mu = M 1_K$ and so the desired result is to find a D such that

$$DADM1_K = 1_K \quad \Longleftrightarrow \quad MDADM1_K = M1_K = \mu \quad \Longleftrightarrow \quad D(MAM)D1_K = \mu.$$

Since MAM has strictly positive entries, it is sufficient to show that for any matrix A (MAM in our case; a slight misuse notation here) with strictly positive entries, there is a unique diagonal matrix D with strictly positive diagonal entries such that

$$DAD1_k = \mu_K. \tag{39}$$

We now show the existence and uniqueness separately.

For existence, we follow the proof in [6]. Consider $d'Ad$ for a vector $d \in \mathbb{R}^K$ with strictly positive entries. It is shown there that $d'Ad$ can be minimized using Lagrange multiplier:

$$\frac{1}{2} d'Ad - \lambda \sum_{k=1}^K \mu_k \log(d_k).$$

Differentiating with respect to d and set the derivative to 0 gives

$$Ad = \lambda \sum_{k=1}^K \mu_k / d_k, \tag{40}$$

where $\lambda = d'Ad/(\sum_{k=1}^K \mu_k) > 0$. Letting $D = \lambda^{-1/2}\mathrm{diag}(d_1, d_2, \ldots, d_K)$. It is seen that (40) can be rewritten as

$$DAD1_K = \mu,$$

and the claim follows.

For uniqueness, we adapt the proof in [5] to our case. Suppose there are two different eligible diagonal matrices D_1 and D_2 satisfying (39). Let $d_1 = D_1 1_K$ and $d_2 = D_2 1_K$, and let $M = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_K)$. It follows that

$$D_2 D_1 A d_1 = D_2 D_1 A D_1 1_K = D_2 \mu = M d_2,$$

and so

$$M^{-1} D_2 D_1 A d_1 = d_2.$$

Now, for a diagonal matrix S with strictly positive diagonal entries to be determined, we have

$$S^{-1} M^{-1} D_2 D_1 A S S^{-1} d_1 = S^{-1} d_2.$$

We pick S such that

$$S^{-1} M^{-1} D_2 D_1 = S,$$

and denote such an S by S_0. It follows

$$S_0 A S_0 (S_0^{-1} d_1) = S_0^{-1} d_2.$$

or equivalently, if we let $\tilde{d}_1 = S_0^{-1} d_1$ and $\tilde{d}_2 = S_0^{-1} d_2$,

$$S_0 A S_0 \tilde{d}_1 = \tilde{d}_2. \tag{41}$$

Similarly, by switching the places of D_1 and D_2, we have

$$S_0 A S_0 \tilde{d}_2 = \tilde{d}_1. \tag{42}$$

Combining (41) and (42) gives

$$S_0 A S_0 (\tilde{d}_1 + \tilde{d}_2) = (\tilde{d}_1 + \tilde{d}_2), \qquad \text{and} \qquad S_0 A S_0 (\tilde{d}_1 - \tilde{d}_2) = -(\tilde{d}_1 - \tilde{d}_2).$$

This implies that 1 and -1 are the two eigenvalues of $S_0 A S_0$, with $\tilde{d}_1 + \tilde{d}_2$ and $\tilde{d}_1 - \tilde{d}_2$ being the corresponding eigenvectors, respectively, where we note that especially, $\tilde{d}_1 + \tilde{d}_2$ has all strictly positive entries. By Perron's theorem [2], since $S_0 A S_0$ have all strictly positive entries, the eigenvector corresponding to the largest eigenvalue (i.e., the Perron root) have all strictly positive entries. As for any symmetric matrix, we can only have one eigenvector that has all strictly positive entries, so 1 must be the Perron root of $S_0 A S_0$. Using Perron's Theorem again, all eigenvalues of $S_0 A S_0$ except the Perron root itself should be strictly smaller than 1 in magnitude. This contradicts with the fact that -1 is an eigenvalue of $S_0 A S_0$. The contradiction proves the uniqueness. □

F.3. Proof of Theorem 3.1. This theorem follows easily from Theorem 3.2 and Theorems 3.3-3.5. Fix (Θ, P, F) such that $\theta \in \mathcal{M}_n^*(\beta_n/2)$ and $\|\theta\| \cdot |\mu_2(P)| \geq 2\alpha_n$. Consider a sequence of hypotheses indexed by n, where $\Omega = \theta\theta'$ under $H_0^{(n)}$, and Ω follows the construction in any of Theorem 3.2 and Theorems 3.3-3.5 under $H_1^{(n)}$. Let $P_0^{(n)}$ and $P_1^{(n)}$ be the probability measures associated with two hypotheses, respectively. By those theorems, the χ^2-distance satisfy

$$\mathcal{D}(P_0^{(n)}, P_1^{(n)}) = o(1), \qquad \text{as } n \to \infty.$$

By connection between L^1-distance and χ^2-distance, it follows that

$$\|P_0^{(n)} - P_1^{(n)}\|_1 = o(1), \qquad \text{as } n \to \infty.$$

We now slightly modify the alternative hypothesis. Let Π_0 be a non-random membership matrix such that $(\theta, \Pi_0, P) \in \mathcal{M}_n(K, c_0, \alpha_n, \beta_n)$. In the modified alternative hypothesis $\widetilde{H}_1^{(n)}$,

$$\Pi = \begin{cases} \widetilde{\Pi}, & \text{if } (\theta, \widetilde{\Pi}, P) \in \mathcal{M}_n(K, c_0, \alpha_n, \beta_n), \\ \Pi_0, & \text{otherwise}, \end{cases} \qquad \text{where} \quad \widetilde{\pi}_i \stackrel{iid}{\sim} F.$$

Let $\widetilde{P}_1^{(n)}$ be the probability measure associated with $\widetilde{H}_1^{(n)}$. By Lemmas F.1-F.2, $\Pi = \widetilde{\Pi}$, except for a vanishing probability. It follows that

$$\|P_1^{(n)} - \widetilde{P}_1^{(n)}\|_1 = o(1), \qquad \text{as } n \to \infty.$$

Under $\widetilde{H}_1^{(n)}$, all realizations (θ, Π, P) are in the class $\mathcal{M}_n(K, c_0, \alpha_n, \beta_n)$. By Neyman-Pearson lemma and elementary inequalities,

$$\begin{aligned} &\inf_{\psi} \Big\{ \sup_{\theta \in \mathcal{M}_n^*(\beta_n)} \mathbb{P}(\psi = 1) + \sup_{(\theta, \Pi, P) \in \mathcal{M}_n(K, c_0, \alpha_n, \beta_n)} \mathbb{P}(\psi = 0) \Big\} \\ &\geq \inf_{\psi} \Big\{ \mathcal{P}_0^{(n)}(\psi = 1) + \widetilde{\mathcal{P}}_1^{(n)}(\psi = 0) \Big\} \\ &\geq 1 - \|P_0^{(n)} - \widetilde{P}_1^{(n)}\|_1 \\ &\geq 1 - \|P_0^{(n)} - P_1^{(n)}\|_1 - \|P_1^{(n)} - \widetilde{P}_1^{(n)}\|_1 \\ &\geq 1 - o(1), \end{aligned}$$

where the second line is because all realizations in $\widetilde{\mathcal{P}}_1^{(n)}$ are in the class $\mathcal{M}_n(K, c_0, \alpha_n, \beta_n)$, and the third line follows from the Neyman-Pearson lemma. □

F.4. Proof of Theorems 3.2-3.5. We note that Theorem 3.2, Theorem 3.4 and Theorem 3.5 can be deduced from Theorem 3.3. To see this, recall that Theorem 3.3 assumes there exists a positive diagonal matrix D such that

$$DPD\widetilde{h}_D = 1_K, \qquad \min_{1 \leq k \leq K} \{\widetilde{h}_{D,k}\} \geq C, \tag{43}$$

where $\widetilde{h}_D = \mathbb{E}[D^{-1}\pi_i / \|D^{-1}\pi_i\|_1]$. We show that the condition (43) is implied by conditions of other theorems. Theorem 3.2 assumes $\pi_i \in \{e_1, e_2, \ldots, e_K\}$. It follows that $D^{-1}\pi_i / \|D^{-1}\pi_i\|_1 = \pi_i$, and so $\widetilde{h}_D = h$. By Lemma 3.1, there exists D such that $DPDh = 1_K$, hence, (43) is satisfied. Theorem 3.4 constructs the alternative hypothesis using $\widetilde{\pi}_i = D\pi_i / \|D\pi_i\|_1$. Equivalently, $D^{-1}\widetilde{\pi}_i / \|D^{-1}\widetilde{\pi}_i\|_1 = \pi_i$, and so $\widetilde{h}_D$ becomes h. Since $DPDh = 1_K$, condition (43) holds. Theorem 3.5 assumes $Ph = q_n 1_K$. Let $D = q_n^{-1/2} I_K$. Then, $\widetilde{h}_D = h$ and $DPDh = q_n^{-1} Ph = 1_K$. Again, (43) is satisfied.

We only need to prove Theorem 3.3. Let $P_0^{(n)}$ and $P_1^{(n)}$ be the probability measure associated with $H_0^{(n)}$ and $H_1^{(n)}$, respectively. Let $\mathcal{D}(P_0^{(n)}, P_1^{(n)})$ be the chi-square distance between two probability measures. By elementary probability,

$$\mathcal{D}(P_0^{(n)}, P_1^{(n)}) = \int \left[\frac{dP_1^{(n)}}{dP_0^{(n)}} \right]^2 dP_0^{(n)} - 1.$$

It suffices to show that, when $\|\theta\| \cdot \mu_2(P) \to 0$,

$$\int \Big[\frac{dP_1^{(n)}}{dP_0^{(n)}}\Big]^2 dP_0^{(n)} = 1 + o(1). \tag{44}$$

Let p_{ij} and $q_{ij}(\Pi)$ be the corresponding Ω_{ij} under the null and the alternative, respectively. It is seen that

$$dP_0^{(n)} = \prod_{i<j} p_{ij}^{A_{ij}}(1-p_{ij})^{1-A_{ij}}, \qquad dP_1^{(n)} = \mathbb{E}_\Pi\Big[\prod_{i<j}[q_{ij}(\Pi)]^{A_{ij}}[1-q_{ij}(\Pi)]^{1-A_{ij}}\Big].$$

Let $\tilde{\Pi}$ be an independent copy of Π. Then,

$$\begin{aligned}\Big[\frac{dP_1^{(n)}}{dP_0^{(n)}}\Big]^2 &= \mathbb{E}_\Pi\Big[\prod_{i<j}\Big(\frac{q_{ij}(\Pi)}{p_{ij}}\Big)^{A_{ij}}\Big(\frac{1-q_{ij}(\Pi)}{1-p_{ij}}\Big)^{1-A_{ij}}\Big] \cdot \mathbb{E}_{\tilde{\Pi}}\Big[\prod_{i<j}\Big(\frac{q_{ij}(\tilde{\Pi})}{p_{ij}}\Big)^{A_{ij}}\Big(\frac{1-q_{ij}(\tilde{\Pi})}{1-p_{ij}}\Big)^{1-A_{ij}}\Big]\\ &= \mathbb{E}_{\Pi,\tilde{\Pi}}\underbrace{\Big[\prod_{i<j}\Big(\frac{q_{ij}(\Pi)q_{ij}(\tilde{\Pi})}{p_{ij}^2}\Big)^{A_{ij}}\Big(\frac{[1-q_{ij}(\Pi)][1-q_{ij}(\tilde{\Pi})]}{[1-p_{ij}]^2}\Big)^{1-A_{ij}}\Big]}_{S(A,\Pi,\tilde{\Pi})}.\end{aligned}$$

It follows that

$$\begin{aligned}\int \Big[\frac{dP_1^{(n)}}{dP_0^{(n)}}\Big]^2 dP_0^{(n)} &= \mathbb{E}_A\Big[\frac{dP_1^{(n)}}{dP_0^{(n)}}\Big]^2\\ &= \mathbb{E}_{A,\Pi,\tilde{\Pi}}[S(A,\Pi,\tilde{\Pi})]\\ &= \mathbb{E}_{\Pi,\tilde{\Pi}}\big\{\mathbb{E}_A\big[S(A,\Pi,\tilde{\Pi})|\Pi,\tilde{\Pi}\big]\big\},\end{aligned}$$

where the distribution of $A|(\Pi,\tilde{\Pi})$ is under the null hypothesis. Under the null hypothesis, A is independent of $(\Pi,\tilde{\Pi})$, the upper triangular entries of A are independent of each other, and $A_{ij} \sim \mathrm{Bernoulli}(p_{ij})$. It follows that

$$\begin{aligned}\mathbb{E}_A\big[S(A,\Pi,\tilde{\Pi})|\Pi,\tilde{\Pi}\big] &= \prod_{i<j}\mathbb{E}_A\Big[\Big(\frac{q_{ij}(\Pi)q_{ij}(\tilde{\Pi})}{p_{ij}^2}\Big)^{A_{ij}}\Big(\frac{[1-q_{ij}(\Pi)][1-q_{ij}(\tilde{\Pi})]}{[1-p_{ij}]^2}\Big)^{1-A_{ij}}\Big|\Pi,\tilde{\Pi}\Big]\\ &= \prod_{i<j}\Big\{p_{ij}\frac{q_{ij}(\Pi)q_{ij}(\tilde{\Pi})}{p_{ij}^2} + (1-p_{ij})\frac{[1-q_{ij}(\Pi)][1-q_{ij}(\tilde{\Pi})]}{[1-p_{ij}]^2}\Big\}\\ &= \prod_{i<j}\Big\{\frac{q_{ij}(\Pi)q_{ij}(\tilde{\Pi})}{p_{ij}} + \frac{[1-q_{ij}(\Pi)][1-q_{ij}(\tilde{\Pi})]}{1-p_{ij}}\Big\}.\end{aligned}$$

Let $\Delta_{ij} = q_{ij}(\Pi) - p_{ij}$ and $\tilde{\Delta}_{ij} = q_{ij}(\tilde{\Pi}) - p_{ij}$. By direct calculations,

$$\frac{q_{ij}(\Pi)q_{ij}(\tilde{\Pi})}{p_{ij}} + \frac{[1-q_{ij}(\Pi)][1-q_{ij}(\tilde{\Pi})]}{1-p_{ij}} = 1 + \frac{\Delta_{ij}\tilde{\Delta}_{ij}}{p_{ij}(1-p_{ij})}.$$

Combining the above gives

$$\int \Big[\frac{dP_1^{(n)}}{dP_0^{(n)}}\Big]^2 dP_0^{(n)} = \mathbb{E}_{\Pi,\tilde{\Pi}}\Big[\prod_{i<j}\Big(1 + \frac{\Delta_{ij}\tilde{\Delta}_{ij}}{p_{ij}(1-p_{ij})}\Big)\Big]. \tag{45}$$

We then plug in the expressions of Δ_{ij} and $\tilde{\Delta}_{ij}$ from the model. Let D be the matrix in (43). Introduce $M = DPD - \mathbf{1}_K\mathbf{1}_K'$. We re-write

$$DPD = \mathbf{1}_K\mathbf{1}_K' + M.$$

It is seen that $M\widetilde{h}_D = \mathbf{0}_K$. The following lemma is proved in Section F.5.

LEMMA F.3. *Under the conditions of Theorem 3.3,* $\|M\| \le C|\mu_2(P)|$.

Write for short $\pi_i^D = \frac{1}{\|D^{-1}\pi_i\|_1}D^{-1}\pi_i$ and $y_i = \pi_i^D - \mathbb{E}[\pi_i^D] = \pi_i^D - \widetilde{h}_D$. Under the alternative hypothesis,

$$\begin{aligned} q_{ij}(\Pi) &= \theta_i\theta_j\|D^{-1}\pi_i\|_1\|D^{-1}\pi_j\|_1 \cdot \pi_i'P\pi_j \\ &= \theta_i\theta_j \cdot (\pi_i^D)'(DPD)(\pi_j^D) \\ &= \theta_i\theta_j \cdot (\pi_i^D)'(\mathbf{1}_K\mathbf{1}_K' + M)(\pi_j^D) \\ &= \theta_i\theta_j \cdot \big[1 + (\pi_i^D)'M(\pi_j^D)\big] \\ &= \theta_i\theta_j \cdot \big[1 + (\widetilde{h}_D + y_i)'M(\widetilde{h}_D + y_j)\big] \\ &= \theta_i\theta_j \cdot (1 + y_i'My_j). \end{aligned}$$

Here, the fourth line is due to $\mathbf{1}_K'\pi_i = 1$ and the last line is due to $M\widetilde{h}_D = \mathbf{0}_K$. Under the null hypothesis, $p_{ij} = \theta_i\theta_j$. As a result,

$$\Delta_{ij} = \theta_i\theta_j \cdot y_i'My_j, \qquad y_i \equiv \pi_i^D - \mathbb{E}[\pi_i^D].$$

Similarly, $\tilde{\Delta}_{ij} = \theta_i\theta_j \cdot \tilde{y}_i'M\tilde{y}_j$, with $\tilde{y}_i = \tilde{\pi}_i^D - \mathbb{E}[\tilde{\pi}_i^D]$. We plug them into (45) and use $p_{ij} = \theta_i\theta_j$. It gives

$$\int \left[\frac{dP_1^{(n)}}{dP_0^{(n)}}\right]^2 dP_0^{(n)} = \mathbb{E}\bigg[\prod_{i<j}\Big(1 + \frac{\theta_i\theta_j}{1-\theta_i\theta_j}(y_i'My_j)(\tilde{y}_i'M\tilde{y}_j)\Big)\bigg], \tag{46}$$

where $\{y_i, \tilde{y}_i\}_{i=1}^n$ are iid random vectors with $\mathbb{E}[y_i] = \mathbf{0}_K$.

We bound the right hand side of (46). Since $1 + x \le e^x$ for all $x \in \mathbb{R}$,

$$\mathcal{D}(P_0^{(n)}, P_1^{(n)}) \le \mathbb{E}[\exp(S)], \qquad \text{where} \quad S \equiv \sum_{i<j}\frac{\theta_i\theta_j}{1-\theta_i\theta_j}(y_i'My_j)(\tilde{y}_i'M\tilde{y}_j).$$

Let $M = \sum_{k=1}^K \delta_k b_k b_k'$ be the eigen-decomposition of M. Then,

$$(y_i'My_j)(\tilde{y}_i'M\tilde{y}_j) = \sum_{1\le k,\ell\le K}\delta_k\delta_\ell(b_k'y_i)(b_k'y_j)(b_\ell'\tilde{y}_i)(b_\ell'\tilde{y}_j).$$

This allows us to decompose

$$S = \frac{1}{K^2}\sum_{1\le k,\ell\le K} S_{k\ell}, \qquad \text{where } S_{k\ell} = K^2\delta_k\delta_\ell\sum_{i<j}\frac{\theta_i\theta_j}{1-\theta_i\theta_j}(b_k'y_i)(b_k'y_j)(b_\ell'\tilde{y}_i)(b_\ell'\tilde{y}_j).$$

By Jensen's inequality, $\exp(\frac{1}{K^2}\sum_{k,\ell}S_{k\ell}) \le \frac{1}{K^2}\sum_{k,\ell}\exp(S_{k\ell})$. It follows that

$$\int \left[\frac{dP_1^{(n)}}{dP_0^{(n)}}\right]^2 dP_0^{(n)} \le \mathbb{E}[\exp(S)] \le \max_{1\le k,\ell\le K}\mathbb{E}[\exp(S_{k\ell})]. \tag{47}$$

We now fix (k,ℓ) and derive a bound for $\mathbb{E}[\exp(S_{k\ell})]$. For n large enough, $\theta_{\max} \leq 1/2$ and $K^4\|M\|^2\|\theta\|^2 \leq 1/9$. By Taylor expansion of $(1-\theta_i\theta_j)^{-1}$,

$$S_{k\ell} = K^2\delta_k\delta_\ell \sum_{i<j}\sum_{m=1}^{\infty} \theta_i^m\theta_j^m (b_k'y_i)(b_k'y_j)(b_\ell'\tilde{y}_i)(b_\ell'\tilde{y}_j)$$

$$\equiv \sum_{m=1}^{\infty} X_m, \qquad \text{where} \quad X_m \equiv K^2\delta_k\delta_\ell \sum_{i<j} \theta_i^m\theta_j^m (b_k'y_i)(b_k'y_j)(b_\ell'\tilde{y}_i)(b_\ell'\tilde{y}_j).$$

Since $|X_m| \leq C\|M\|^2\|\theta\|_m^{2m} \leq C\|M\|\|\theta\|_1^2\theta_{\max}^{2(m-1)}$, where $\sum_{m=1}^{\infty}\theta_{\max}^{2(m-1)} < \infty$, the random variable $\sum_{m=1}^{\infty} X_m$ is always well-defined. For $m \geq 1$, let $a_m = \theta_{\max}^{2(m-1)}(1-\theta_{\max}^2)$. Then, $\sum_{m=1}^{\infty} a_m = 1$. By Jenson's inequality,

$$\exp\Big(\sum_{m=1}^{\infty} X_m\Big) = \exp\Big(\sum_{m=1}^{\infty} a_m \cdot a_m^{-1}|X_m|\Big) \leq \sum_{m=1}^{\infty} a_m \cdot \exp(a_m^{-1}X_m).$$

Using Fatou's lemma, we have

$$\mathbb{E}[\exp(S_{k\ell})] \leq \sum_{m=1}^{\infty} a_m \cdot \mathbb{E}\big[\exp(a_m^{-1}X_m)\big]. \tag{48}$$

By definition of X_m,

$$X_m = K^2\delta_k\delta_\ell \bigg\{ \Big[\sum_i \theta_i^m (b_k'y_i)(b_\ell'\tilde{y}_i)\Big]^2 - \sum_i \theta_i^{2m}(b_k'y_i)^2(b_\ell'\tilde{y}_i)^2 \bigg\}.$$

Note that $\max_i\{\|y_i\|, \|\tilde{y}_i\|\} \leq \sqrt{K}$ and $\max_k|\delta_k| = \|M\|$. Therefore,

$$|X_m| \leq K^2\|M\|^2\Big[\sum_i \theta_i^m (b_k'y_i)(b_\ell'\tilde{y}_i)\Big]^2 + K^4\|M\|^2\|\theta\|_{2m}^{2m}.$$

Write $Y = \sum_i \theta_i^m (b_k'y_i)(b_\ell'\tilde{y}_i)$. We see that Y is sum of independent, mean-zero random variables. Since $|(b_k'y_i)(b_\ell'\tilde{y}_i)| \leq K$, by Hoeffding's inequality,

$$\mathbb{P}(|Y| > t) \leq 2\exp\Big(-\frac{t^2}{4K^2\|\theta\|_{2m}^{2m}}\Big), \qquad \text{for any } t > 0.$$

Since $\|\theta\|_{2m}^{2m} \leq \|\theta\|^2\theta_{\max}^{2(m-1)} \leq 2a_m\|\theta\|^2$, we have $a_m^{-1}K^4\|M\|^2\|\theta\|_{2m}^{2m} \leq 2K^4\|M\|^2\|\theta\|^2$. Note that $K^4\|M\|^2\|\theta\|^2 \leq 1/9$. By direct calculations,

$$\begin{aligned}
\mathbb{E}\big[\exp(a_m^{-1}|X_m|)\big] &\leq e^{a_m^{-1}K^4\|M\|^2\|\theta\|_{2m}^{2m}} \cdot \mathbb{E}\big[e^{a_m^{-1}K^2\|M\|^2Y^2}\big] \\
&\leq e^{2K^4\|M\|^2\|\theta\|^2} \cdot \mathbb{E}\big[e^{a_m^{-1}K^2\|M\|^2Y^2}\big] \\
&= e^{2K^4\|M\|^2\|\theta\|^2}\Big[1 + \int_0^{\infty} e^t \cdot \mathbb{P}\big(a_m^{-1}K^2\|M\|^2Y^2 > t\big)dt\Big] \\
&\leq e^{2K^4\|M\|^2\|\theta\|^2}\Big[1 + \int_0^{\infty} e^t \cdot e^{-\frac{t}{8K^4\|M\|^2\|\theta\|^2}}dt\Big] \\
&\leq e^{K^4\|M\|^2\|\theta\|^2} \cdot (1 + 72K^4\|M\|^2\|\theta\|^2).
\end{aligned}$$

We plug it into (48) and notice that $\sum_{m=1}^{\infty} a_m = 1$. It gives

$$\mathbb{E}[\exp(S_{k\ell})] \leq e^{K^4\|M\|^2\|\theta\|^2} \cdot (1 + 72K^4\|M\|^2\|\theta\|^2). \tag{49}$$

Combining (47) and (49) gives

$$\int \left[\frac{dP_1^{(n)}}{dP_0^{(n)}}\right]^2 dP_0^{(n)} \le e^{K^4\|M\|^2\|\theta\|^2}\cdot(1+72K^4\|M\|^2\|\theta\|^2).$$

We recall that $\|\theta\|\cdot\|M\| \le C\|\theta\|\cdot|\mu_2(P)| \to 0$. Hence, the right hand side is $1+o(1)$. This proves (44).

F.5. Proof of Lemmas F.1-F.3.

F.5.1. *Proof of Lemma F.1.* The first claim follows by our assumptions on P, so we omit the proof. Consider the second claim. Recall that $G = \|\theta\|^{-2}\Pi'\Theta^2\Pi$ and $d_1, d_2, \ldots, d_K$ are the eigenvalues of $G^{1/2}PG^{1/2}$, arranged in the descending order in magnitude. By Lemmas D.1 and D.2, $\lambda_k = \|\theta\|^2 d_k$, $1 \le k \le K$, and $d_1 \asymp 1$. Combining these, it suffices to show

$$|\mu_2| \asymp |d_2|.$$

We now prove for the cases where P is non-singular and singular, separately. Consider the first case. Since $1/d_k$ and $1/\mu_K$ are the largest eigenvalue of $G^{-1/2}P^{-1/2}G^{-1/2}$ and P^{-1} in magnitude, respectively, and $\|G\| \le C$ and $\|G^{-1}\| \le C$, it is seen that $|\mu_K| \asymp |d_K|$. To show the claim, it sufficient to show that for any $m \ge 2$, if $|\mu_k| \asymp |d_k|$ for $k = m+1, \ldots, K$, then $|\mu_m| \asymp |d_m|$.

We now fix $m \ge 2$, and assume $|\mu_k| \asymp |d_k|$ for $k = m+1, \ldots, K$. The goal is to show $|\mu_m| \asymp |d_m|$. By symmetry, it is sufficient to show that

$$|d_m| \le C|\mu_m|. \tag{50}$$

Let $P = V\mathrm{diag}(d_1, d_2, \ldots, d_K)V'$ be the SVD of P, where $V \in \mathbb{R}^{K,K}$ is orthonormal, and let V_m be the sub-matrix of V consisting the first m columns of V. Introduce

$$\widetilde{P}_m = V_m D_m V_m', \qquad \text{where } D_m = \mathrm{diag}(d_1, d_2, \ldots, d_m).$$

Let $\mu_1^*, \mu_2^*, \ldots, \mu_m^*$ and $d_1^*, d_2^*, \ldots, d_m^*$ be the first m eigenvalues of $\widetilde{P}_m$ and $G^{1/2}P_mG^{1/2}$, respectively, arranged in the descending order in magnitude. Since $\|G\| \le C$, we have

$$\|P - P_m\| \le C|\mu_{m+1}|, \qquad \|G^{1/2}(P-P_m)G^{1/2}\| \le C|\mu_{m+1}|.$$

By Theorem [1, Theorem A.46],

$$|\mu_m - \mu_m^*| \le C\|P - P_m\| \le |\lambda_{m+1}|, \tag{51}$$

and

$$|d_m - d_m^*| \le \|G^{1/2}(P-P_m)G^{1/2}\| \le C|\mu_{m+1}|. \tag{52}$$

At the same time, note that the nonzero eigenvalues of $G^{1/2}P_mG^{1/2}$ are the same as the nonzero eigenvalues of $D_mV_m'GV_m$, and also the same as those of $(V_m'GV_m)^{1/2}D_m(V_m'GV_m)^{1/2}$. Since $\|G\| \le C$ and $\|G^{-1}\| \le C$, it is seen $\|V_m'GV_m\| \le C$ and $\|V_m'GV_m)^{-1}\| \le C$. Therefore, by similar arguments,

$$|\mu_m^*| \asymp |d_m^*|. \tag{53}$$

Combining (51), (52), and (53) gives

$$\begin{aligned}|\mu_m| \le |\mu_m^*| + |\mu_m - \mu_m^*| &\le C(|d_m^*| + |d_{m+1}|)\\ \le C[(|d_m| + |d_m - d_m^*|) + |d_{m+1}|] &\le C|d_m|.\end{aligned}$$

This proves (50) and the claim follows.

We now consider the case where P is singular, say, $rank(P) = r < K$, and the nonzero eigenvalues are $\mu_1, \mu_2, \ldots, \mu_r$. Let $P = UDU'$ be the SVD, where $U \in \mathbb{R}^{n,r}$ and $D = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_r)$. By similar argument, the nonzero eigenvalues of $G^{1/2}PG^{1/2}$ are the same as $(U'GU)^{1/2}D(U'GU)^{1/2}$, where $\|U'GU\| \leq C$ and $\|(U'GU)^{-1}\| \leq C$. The remaining part of the proof is similar so is omitted.

Consider the last claim. Let $\widetilde{P} = \eta\eta'$, where η is the first eigenvector of P, scaled to have a ℓ^2-norm of $\sqrt{\mu_1}$. Write

$$|P_{ij} - 1| = |P_{ij} - \eta_i\eta_j| + |\eta_i\eta_j - 1|. \tag{54}$$

Now, first, by definitions and elementary algebra, for $1 \leq i, j \leq K$,

$$|P_{ij} - \eta_i\eta_j| \leq |P_{ij} - \widetilde{P}_{ij}| \leq \|P - \widetilde{P}\| \leq \mu_2, \tag{55}$$

where by the second claim, $\mu_2 = o(1)$. Note that for $1 \leq i, j \leq K$, $P_{ii} = 1$ and $P_{ij} \geq 0$. It is seen that $|\eta_i| = 1 + o(1)$ and all η_i must have the positive sign. It follows $|\eta_i - 1| = (1 + \eta_i)^{-1}(1 - \eta_i^2) \leq \mu_2$, and so

$$|1 - \eta_i\eta_j| \leq |(1 - \eta_i)(1 - \eta_j)| + |1 - \eta_i| + |1 - \eta_j| \leq C\mu_2. \tag{56}$$

Combining (54)-(56) gives the claim. □

F.5.2. *Proof of Lemma F.2.* Consider the first claim about $\sum_i \theta_i\pi_i(k)$. Write $X = \sum_{i=1}^n \theta_i(\pi_i(k) - h_k)$. It is seen that X is sum of independent mean-zero random variables, where $\theta_i|\pi_i(k) - h_k| \leq C\theta_{\max}$ and $\sum_{i=1}^n \mathrm{Var}(\theta_i(\pi_i(k) - h_k)) \leq C\|\theta\|^2$. By Bernstein's inequality, for any $t > 0$,

$$\mathbb{P}(|X| > t) \leq \exp\Big(-\frac{t^2}{C\|\theta\|^2 + C\theta_{\max}t}\Big).$$

It follows that, with probability $1 - \|\theta\|_1^{-1}$,

$$\Big|\sum_i \theta_i\pi_i(k) - h_k\|\theta\|_1\Big| = |X| \leq C\|\theta\|\sqrt{\log(\|\theta\|_1)} + C\theta_{\max}\log(\|\theta\|_1).$$

Since $\|\theta\| \to \infty$, $\theta_{\max} \to 0$, and $(\|\theta\|^2/\|\theta\|_1)\sqrt{\log(\|\theta\|_1)} \to 0$, the right hand side is $o(\|\theta\|_1)$. Combining it with the assumption of $\min_k\{h_k\} \geq C$, we have

$$\sum_i \theta_i\pi_i(k) \geq C\|\theta\|_1, \qquad \text{with probability } 1 - \|\theta\|^{-1} = 1 - o(1).$$

Additionally, since $\pi_i(k) \leq 1$, $\sum_i \theta_i\pi_i(k) \leq \|\theta\|_1$. Therefore, with probability $1 - o(1)$, each $\sum_i \theta_i\pi_i(k)$ is at the order of $\|\theta\|_1$. This proves the first claim.

Consider the second claim about G. Let $y_i = \pi_i - h$. Then, $\pi_i\pi_i' = hh' + hy_i' + y_ih' + y_iy_i'$ and $\Sigma = \mathbb{E}[\pi_i\pi_i'] = hh' + \mathbb{E}[y_iy_i']$. It follows that

$$\begin{aligned}
\|\theta\|^2 G = \sum_{i=1}^n \theta_i^2\pi_i\pi_i' &= \sum_{i=1}^n \theta_i^2\big(\Sigma + hy_i' + y_ih' + y_iy_i' - \mathbb{E}[y_iy_i']\big) \\
&= \|\theta\|^2\Sigma + \sum_{i=1}^n \theta_i^2(y_iy_i' - \mathbb{E}[y_iy_i']) + \sum_{i=1}^n \theta_i^2 hy_i' + \sum_{i=1}^n \theta_i^2 y_ih' \\
&\equiv \|\theta\|^2\Sigma + Z_0 + Z_1 + Z_2.
\end{aligned}$$

Here, Z_0 is the sum of independent, mean-zero random matrices. We apply the matrix Hoeffding inequality [7] to bound its operator norm. Since $\theta_i^2\|y_iy_i' - \mathbb{E}[y_iy_i']\| \leq C\theta_i^2$, the matrix

Hoeffding inequality implies that $\mathbb{P}\big(\|Z_0\| > t\big) \le \exp\big(-\frac{t^2}{C^*\|\theta\|_4^4}\big)$ for all $t > 0$, where $C^* > 0$ is a constant. Let ζ_n be a sequence such that $\zeta_n \to \infty$. With $t = \|\theta\|_4^2\sqrt{C^*\log(\zeta_n)}$, we have

$$\|Z_0\| \le C\|\theta\|_4^2\sqrt{\log(\zeta_n)}, \qquad \text{with probability } 1-\zeta_n.$$

Similarly, we can apply the matrix Hoeffding inequality to Z_1 and Z_2. It gives

$$\|Z_1 + Z_2\| \le C\|\theta\|_4^2\sqrt{\log(\zeta_n)}, \qquad \text{with probability } 1-\zeta_n.$$

Since $\|\theta\|_4^2 \le \theta_{\max}\|\theta\| \ll \|\theta\|^2$, we can choose ζ_n so that $\|\theta\|_4^2\sqrt{\log(\zeta_n)} = o(\|\theta\|^2)$. It follows that, with probability $1-o(1)$,

$$\|Z_0 + Z_1 + Z_2\| = o(\|\theta\|^2).$$

At the same time, $\lambda_{\min}(\|\theta\|^2\Sigma) = \|\theta\|^2\,\|\Sigma^{-1}\|^{-1} \ge C\|\theta\|^2$. Therefore, with probability $1-o(1)$,

$$\lambda_{\min}(\|\theta\|^2 G) \ge \lambda_{\min}(\|\theta\|^2\Sigma) - \|Z_0 + Z_1 + Z_2\| \ge C\|\theta\|^2.$$

This guarantees $\|G^{-1}\| \le C$. □

F.5.3. *Proof of Lemma F.3*. Let $Q = P - 1_K 1_K'$, and introduce $d \in \mathbb{R}^K$ such that $D = \mathrm{diag}(d)$. By Lemma F.1, $\|Q\| \le C|\mu_2|$. With these notations,

$$DPD - 1_K 1_K' = dd' + DQD - 1_K 1_K'. \tag{57}$$

Using the same notations, the assumption $DPD\widetilde{h}_D = 1_K$ can be written as $D(1_K 1_K' + Q)D\widetilde{h}_D = 1_K$. It implies

$$1_K = (d'\widetilde{h}_D)d + DQD\widetilde{h}_D. \tag{58}$$

We multiply $\widetilde{h}_D'$ on both sides and notice that $1_K'\widetilde{h}_D = 1$. It gives

$$(d'\widetilde{h}_D)^2 = 1 - \widetilde{h}_D' DQD\widetilde{h}_D. \tag{59}$$

Combining (58)-(59) gives

$$\begin{aligned} dd' - 1_K 1_K' &= [1 - (d'\widetilde{h}_D)^2]dd' - (d'\widetilde{h}_D)(DQD\widetilde{h}_D d + d\widetilde{h}_D DQD) - DQD\widetilde{h}_D\widetilde{h}_D' DQD \\ &= (\widetilde{h}_D' DQD\widetilde{h}_D)\cdot dd' - (d'\widetilde{h}_D)(DQD\widetilde{h}_D d + d\widetilde{h}_D DQD) - DQD\widetilde{h}_D\widetilde{h}_D' DQD. \end{aligned}$$

Since $\|\widetilde{h}_D\| \le C$ and $\|d\| \le C$, we immediately have

$$\|dd' - 1_K 1_K'\| \le C\|Q\| \le C|\mu_2|.$$

Plugging it into (57) gives

$$\|DPD - 1_K 1_K'\| \le C\|Q\| \le C|\mu_2|.$$

□

APPENDIX G: PROPERTIES OF SIGNED POLYGON STATISTICS

We prove Tables A.1-2 and Theorem A.1-4.3. The analysis of T_n and Q_n is very similar. To save space, we only present the proof for results of Q_n. The proof for results of T_n (Tables A.1, A.2, and Theorems A.1, A.2, A.3) is omitted.

We recall the following notations:

$$\widetilde{\Omega} = \Omega - (\eta^*)(\eta^*)', \qquad \text{where} \quad \eta^* = \frac{1}{\sqrt{v_0}}\Omega \mathbf{1}_n, \;\; v_0 = \mathbf{1}_n'\Omega\mathbf{1}_n;$$

$$\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i), \qquad \text{where} \quad \eta = \frac{1}{\sqrt{v}}(\mathbb{E}A)\mathbf{1}_n, \;\; \tilde{\eta} = \frac{1}{\sqrt{v}}A\mathbf{1}_n, \;\; v = \mathbf{1}_n'(\mathbb{E}A)\mathbf{1}_n;$$

$$r_{ij} = (\eta_i^*\eta_j^* - \eta_i\eta_j) - (\eta_i - \tilde{\eta}_i)(\eta_j - \tilde{\eta}_j) + (1 - \frac{v}{V})\tilde{\eta}_i\tilde{\eta}_j, \qquad \text{where } V = \mathbf{1}_n'A\mathbf{1}_n.$$

Then, the Ideal SgnQ statistic equals to

$$\widetilde{Q}_n = \sum_{i,j,k,\ell(dist)} (\widetilde{\Omega}_{ij} + W_{ij})(\widetilde{\Omega}_{jk} + W_{jk})(\widetilde{\Omega}_{k\ell} + W_{k\ell})(\widetilde{\Omega}_{\ell i} + W_{\ell i}),$$

the Proxy SgnQ statistic equals to

$$Q_n^* = \sum_{i,j,k,\ell(dist)} (\widetilde{\Omega}_{ij} + W_{ij} + \delta_{ij})(\widetilde{\Omega}_{jk} + W_{jk} + \delta_{jk})(\widetilde{\Omega}_{k\ell} + W_{k\ell} + \delta_{k\ell})(\widetilde{\Omega}_{\ell i} + W_{\ell i} + \delta_{\ell i}),$$

and the SgnQ statistic equals to

$$Q_n = \sum_{i,j,k,\ell(dist)} (\widetilde{\Omega}_{ij} + W_{ij} + \delta_{ij} + r_{ij})(\widetilde{\Omega}_{jk} + W_{jk} + \delta_{jk} + r_{jk})(\widetilde{\Omega}_{k\ell} + W_{k\ell} + \delta_{k\ell} + r_{k\ell})(\widetilde{\Omega}_{\ell i} + W_{\ell i} + \delta_{\ell i} + r_{\ell i}).$$

As explained in Section 4, each of $\widetilde{Q}_n, Q_n^*, Q_n$ is the sum of a finite number of post-expansion sums, each having the form

$$\sum_{i,j,k,\ell(dist)} a_{ij}b_{jk}c_{k\ell}d_{\ell i}, \tag{60}$$

where a_{ij} equals to one of $\{\widetilde{\Omega}_{ij}, W_{ij}, \delta_{ij}, r_{ij}\}$; same for b_{ij}, c_{ij} and d_{ij}. Let $N_{\widetilde{\Omega}}$ be the (common) number of $\widetilde{\Omega}$ terms in each product; similarly, we define N_W, N_δ, N_r. These numbers satisfy $N_{\widetilde{\Omega}} + N_W + N_\delta + N_r = 4$. For example, for the post-expansion sum $\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}W_{jk}W_{k\ell}W_{\ell i}$, $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_r) = (1, 3, 0, 0)$. In Section G.1, we study $\widetilde{Q}_n$, and it involves these post-expansion sums such that

$$N_\delta = N_r = 0,$$

In Section G.2, we study $(Q_n^* - \widetilde{Q}_n)$, which involves post-expansion sums such that

$$N_\delta > 0, \;\; \text{and} \;\; N_r = 0,$$

In Section G.3, we study $(Q_n - Q_n^*)$, which is related to the sums such that

$$N_r > 0.$$

G.1. Analysis of Table 1, proof of Theorem 4.1.

Define

$$X_1 = \sum_{i,j,k,\ell(dist)} W_{ij}W_{jk}W_{k\ell}W_{\ell i}, \qquad X_2 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}W_{jk}W_{k\ell}W_{\ell i},$$

$$X_3 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i}, \qquad X_4 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}W_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i},$$

$$X_5 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i}, \qquad X_6 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}.$$

We first consider the null hypothesis. Since $\widetilde{\Omega}$ is a zero matrix, it is not hard to see that

$$\widetilde{Q}_n = X_1.$$

The following lemmas are proved in Section G.4.

LEMMA G.1. *Suppose the conditions of Theorem 4.1 hold. Under the null hypothesis, as $n \to \infty$, $\mathbb{E}[\widetilde{Q}_n] = 0$ and $\mathrm{Var}(\widetilde{Q}_n) = 8\|\theta\|^8 \cdot [1 + o(1)]$.*

LEMMA G.2. *Suppose the conditions of Theorem 4.1 hold. Under the null hypothesis, as $n \to \infty$,*

$$\frac{\widetilde{Q}_n - E[\widetilde{Q}_n]}{\sqrt{\mathrm{Var}(\widetilde{Q}_n)}} \longrightarrow N(0,1), \qquad \textit{in law}.$$

We then consider the alternative hypothesis. By elementary algebra,

$$\widetilde{Q}_n = X_1 + 4X_2 + 4X_3 + 2X_4 + 4X_5 + X_6.$$

The following lemma characterizes the asymptotic mean and variance of X_1-X_6 under the alternative hypothesis. It gives rise to Columns 5-6 of Table 1.

LEMMA G.3 (Table 1). *Suppose conditions of Theorem 4.1 hold. Write $\alpha = |\lambda_2|/\lambda_1$. Under the alternative hypothesis, as $n \to \infty$,*

- $\mathbb{E}[X_k] = 0$ *for* $1 \le k \le 5$, *and* $\mathbb{E}[X_6] = \mathrm{tr}(\widetilde{\Omega}^4) \cdot [1 + o(1)]$.
- $C^{-1}\|\theta\|^8 \le \mathrm{Var}(X_1) \le C\|\theta\|^8$.
- $\mathrm{Var}(X_2) \le C\alpha^2\|\theta\|^4\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $\mathrm{Var}(X_3) \le C\alpha^4\|\theta\|^6\|\theta\|_3^6 = o(\alpha^6\|\theta\|^8\|\theta\|_3^6)$.
- $\mathrm{Var}(X_4) \le C\alpha^4\|\theta\|_3^{12} = o(\|\theta\|^8)$.
- $\mathrm{Var}(X_5) \le C\alpha^6\|\theta\|^8\|\theta\|_3^6$.

As a result, $\mathbb{E}[\widetilde{Q}_n] \sim \mathrm{tr}(\widetilde{\Omega}^4)$ *and* $\mathrm{Var}(\widetilde{Q}_n) \le C(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6)$.

Theorem 4.1 follows directly from Lemmas G.1-G.3.

G.2. Analysis of Table 2, proof of Theorem 4.2. We introduce U_a, U_b and U_c such that

$$Q_n^* - \widetilde{Q}_n = U_a + U_b + U_c,$$

where U_a, U_b and U_c contain post-expansion sums (60) with $N_\delta = 1$, $N_\delta = 2$, and $N_\delta \ge 3$, respectively.

First, we consider the post-expansion sums with $N_\delta = 1$. Define

$$U_a = 4Y_1 + 8Y_2 + 4Y_3 + 8Y_4 + 4Y_5 + 4Y_6, \tag{61}$$

where

$$Y_1 = \sum_{i,j,k,\ell (dist)} \delta_{ij} W_{jk} W_{k\ell} W_{\ell i}, \qquad Y_2 = \sum_{i,j,k,\ell (dist)} \delta_{ij} \widetilde{\Omega}_{jk} W_{k\ell} W_{\ell i},$$

$$Y_3 = \sum_{i,j,k,\ell (dist)} \delta_{ij} W_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i}, \qquad Y_4 = \sum_{i,j,k,\ell (dist)} \delta_{ij} \widetilde{\Omega}_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i},$$

$$Y_5 = \sum_{i,j,k,\ell (dist)} \delta_{ij} \widetilde{\Omega}_{jk} W_{k\ell} \widetilde{\Omega}_{\ell i}, \qquad Y_6 = \sum_{i,j,k,\ell (dist)} \delta_{ij} \widetilde{\Omega}_{jk} \widetilde{\Omega}_{k\ell} \widetilde{\Omega}_{\ell i}.$$

Under the null hypothesis, only Y_1 is nonzero, and

$$U_a = 4Y_1.$$

LEMMA G.4. *Suppose the conditions of Theorem 4.1 hold. Under the null hypothesis, as* $n \to \infty$, $\mathbb{E}[U_a] = 0$ *and* $\mathrm{Var}(U_a) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8)$.

Under the alternative hypothesis, the following lemma characterizes the asymptotic means and variances of Y_1-Y_6. It gives rise to Rows 1-6 of Table 2 and is proved in Section G.4.

LEMMA G.5 (Table 2, Rows 1-6). *Suppose the conditions of Theorem 4.1 hold. Let* $\alpha = |\lambda_2|/\lambda_1$. *Under the alternative hypothesis, as* $n \to \infty$,

- $\mathbb{E}[Y_k] = 0$ *for* $k \in \{1, 2, 3, 5, 6\}$, *and* $|\mathbb{E}[Y_4]| \le C\alpha^2\|\theta\|^6 = o(\alpha^4\|\theta\|^8)$.
- $\mathrm{Var}(Y_1) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $\mathrm{Var}(Y_2) \le C\alpha^2\|\theta\|^4\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $\mathrm{Var}(Y_3) \le C\alpha^2\|\theta\|^4\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $\mathrm{Var}(Y_4) \le \frac{C\alpha^4\|\theta\|^{10}\|\theta\|_3^3}{\|\theta\|_1} = o(\alpha^6\|\theta\|^8\|\theta\|_3^6)$.
- $\mathrm{Var}(Y_5) \le \frac{C\alpha^4\|\theta\|^4\|\theta\|_3^9}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $\mathrm{Var}(Y_6) \le \frac{C\alpha^6\|\theta\|^{12}\|\theta\|_3^3}{\|\theta\|_1} = O(\alpha^6\|\theta\|^8\|\theta\|_3^6)$.

As a result, $\mathbb{E}[U_a] = o(\alpha^4\|\theta\|^8)$ *and* $\mathrm{Var}(U_a) \le C\alpha^6\|\theta\|^8\|\theta\|_3^6 + o(\|\theta\|^8)$.

Next, we consider the post-expansion sums with $N_\delta = 2$. Define

$$U_b = 4Z_1 + 2Z_2 + 8Z_3 + 4Z_4 + 4Z_5 + 2Z_6, \tag{62}$$

where

$$Z_1 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}W_{k\ell}W_{\ell i}, \qquad Z_2 = \sum_{i,j,k,\ell (dist)} \delta_{ij}W_{jk}\delta_{k\ell}W_{\ell i},$$
$$Z_3 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i}, \qquad Z_4 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\widetilde{\Omega}_{jk}\delta_{k\ell}W_{\ell i},$$
$$Z_5 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}, \qquad Z_6 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\widetilde{\Omega}_{jk}\delta_{k\ell}\widetilde{\Omega}_{\ell i}.$$

Under the null hypothesis, only Z_1 and Z_2 are nonzero, and

$$U_b = 4Z_1 + 2Z_2.$$

LEMMA G.6. *Suppose the conditions of Theorem 4.1 hold. Under the null hypothesis, as* $n \to \infty$,

- $\mathbb{E}[Z_1] = \|\theta\|^4 \cdot [1 + o(1)]$, *and* $\mathrm{Var}(Z_1) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $\mathbb{E}[Z_2] = 2\|\theta\|^4 \cdot [1 + o(1)]$, *and* $\mathrm{Var}(Z_2) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.

As a result, $\mathbb{E}[U_b] \sim 8\|\theta\|^4$ *and* $\mathrm{Var}(U_b) = o(\|\theta\|^8)$.

Under the alternative hypothesis, the following lemma provides the asymptotic means and variances of Z_1-Z_6. It gives rise to Rows 7-12 of Table 2:

LEMMA G.7 (Table 2, Rows 7-12). *Suppose conditions of Theorem 4.1 hold. Write* $\alpha = |\lambda_2|/\lambda_1$. *Under the alternative hypothesis, as* $n \to \infty$,

- $|\mathbb{E}[Z_1]| \le C\|\theta\|^4 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(Z_1) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $|\mathbb{E}[Z_2]| \le C\|\theta\|^4 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(Z_2) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $\mathbb{E}Z_3 = 0$, *and* $\mathrm{Var}(Z_3) \le C\alpha^2\|\theta\|^4\|\theta\|_3^6 = o(\|\theta\|^8)$.
- $|\mathbb{E}[Z_4]| \le C\alpha\|\theta\|^4 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(Z_4) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $|\mathbb{E}[Z_5]| \le C\alpha^2\|\theta\|^6 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(Z_5) \le \frac{C\alpha^4\|\theta\|^{14}}{\|\theta\|_1^2} = o(\alpha^6\|\theta\|^8\|\theta\|_3^6)$.
- $|\mathbb{E}[Z_6]| \le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^2} = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(Z_6) \le \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^2} = o(\|\theta\|^8)$.

As a result, $\mathbb{E}[U_b] = o(\alpha^4\|\theta\|^8)$ *and* $\mathrm{Var}(U_b) = o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6)$.

Last, we consider the post-expansion sums with $N_\delta \ge 3$. Define

$$U_c = 4T_1 + 4T_2 + F, \tag{63}$$

where

$$T_1 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\delta_{k\ell}W_{\ell i}, \qquad T_2 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\delta_{k\ell}\widetilde{\Omega}_{\ell i},$$
$$F = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\delta_{k\ell}\delta_{\ell i}.$$

Under the null hypothesis, only T_1 and F are nonzero, and

$$U_b = 4T_1 + F.$$

LEMMA G.8. *Suppose the conditions of Theorem 4.1 hold. Under the null hypothesis, as* $n \to \infty$,

- $\mathbb{E}[T_1] = -2\|\theta\|^4 \cdot [1 + o(1)]$, *and* $\mathrm{Var}(T_1) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $|\mathbb{E}[F]| = 2\|\theta\|^4 \cdot [1 + o(1)]$, *and* $\mathrm{Var}(F) \le \frac{C\|\theta\|^{10}}{\|\theta\|_1^2} = o(\|\theta\|^8)$.

As a result, $\mathbb{E}[U_c] \sim -6\|\theta\|^4$ *and* $\mathrm{Var}(U_c) = o(\|\theta\|^8)$.

Under the alternative hypothesis, the next lemma studies the asymptotic means and variances of T_1, T_2 and F. It gives rise to Rows 13-15 of Table 2:

LEMMA G.9 (Table 2, Rows 13-15). *Suppose conditions of Theorem 4.1 hold. Write* $\alpha = |\lambda_2|/\lambda_1$. *Under the alternative hypothesis, as* $n \to \infty$,

- $|\mathbb{E}[T_1]| \le C\|\theta\|^4 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(T_1) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $|\mathbb{E}[T_2]| \le \frac{C\alpha\|\theta\|^6}{\|\theta\|_1^3} = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(T_2) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8)$.
- $|\mathbb{E}[F]| \le C\|\theta\|^4 = o(\alpha^4\|\theta\|^8)$, *and* $\mathrm{Var}(F) \le \frac{C\|\theta\|^{10}}{\|\theta\|_1^2} = o(\|\theta\|^8)$.

As a result, $\mathbb{E}|U_c| = o(\alpha^4\|\theta\|^8)$ *and* $\mathrm{Var}(U_c) = o(\|\theta\|^8)$.

We now prove Theorem 4.2. Since $Q_n^* - \widetilde{Q}_n = U_a + U_b + U_c$, we have

$$\mathbb{E}[Q_n^* - \widetilde{Q}_n] = \mathbb{E}[U_a] + \mathbb{E}[U_b] + \mathbb{E}[U_c],$$
$$\mathrm{Var}(Q_n^* - \widetilde{Q}_n) \le 3\mathrm{Var}(U_a) + 3\mathrm{Var}(U_b) + 3\mathrm{Var}(U_c).$$

Consider the null hypothesis. By Lemmas G.4, G.6, G.8,

$$\mathbb{E}[Q_n^* - \widetilde{Q}_n] = 0 + 8\|\theta\|^4 - 6\|\theta\|^4 + o(\|\theta\|^4) \sim 2\|\theta\|^4,$$

and

$$\mathrm{Var}(Q_n^* - \widetilde{Q}_n) \le C\|\theta\|^2\|\theta\|_3^6 + \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} + \frac{C\|\theta\|^{10}}{\|\theta\|_1^2}.$$

Using the universal inequality $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$, we further have

$$\mathrm{Var}(Q_n^* - \widetilde{Q}_n) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8),$$

where $\|\theta\|_3^3 = o(\|\theta\|^2)$ and $\|\theta\| \to \infty$ in our range of interest. This proves claims for the null hypothesis. Consider the alternative hypothesis. By Lemmas G.5, G.7, G.9,

$$\big|\mathbb{E}[Q_n^* - \widetilde{Q}_n]\big| \le C\alpha^2\|\theta\|^6,$$

where the main contributors are Y_4 and Z_5. Since $\alpha\|\theta\| \to \infty$ in our range of interest, the above is $o(\alpha^4\|\theta\|^8)$. By Lemmas G.5, G.7, G.9,

$$\mathrm{Var}(Q_n^* - \widetilde{Q}_n) \le \frac{C\alpha^6\|\theta\|^{12}\|\theta\|_3^3}{\|\theta\|_1},$$

where the main contributor is Y_6. Using the universal inequality of $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$, the above is $O(\alpha^6\|\theta\|^8\|\theta\|_3^6)$. This proves claims for the alternative hypothesis.

G.3. Analysis of $(Q_n - Q_n^*)$, proof of Theorem 4.3. By definition, $(Q_n - Q_n^*)$ expands to the sum of 175 post-expansion sums, where each has the form (60) and satisfies $N_r > 0$. Recall that

$$r_{ij} = (\eta_i^*\eta_j^* - \eta_i\eta_j) - (\eta_i - \tilde{\eta}_i)(\eta_j - \tilde{\eta}_j) + (1 - \frac{v}{V})\tilde{\eta}_i\tilde{\eta}_j.$$

Since $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$, we have $\tilde{\eta}_i\tilde{\eta}_j = \eta_i\eta_j - \delta_{ij} + (\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)$. Inserting it into the definition of r_{ij} gives

$$r_{ij} = (\eta_i^*\eta_j^* - \eta_i\eta_j) + (1 - \frac{v}{V})\eta_i\eta_j - (1 - \frac{v}{V})\delta_{ij} - \frac{v}{V}(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j). \tag{64}$$

Define

$$\tilde{r}_{ij} = -\frac{v}{V}(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j), \qquad \epsilon_{ij} = (\eta_i^*\eta_j^* - \eta_i\eta_j) + (1 - \frac{v}{V})\eta_i\eta_j - (1 - \frac{v}{V})\delta_{ij}.$$

Then, we can write

$$r_{ij} = \tilde{r}_{ij} + \epsilon_{ij}. \tag{65}$$

Using this notation, we re-write

$$Q_n = \sum_{i,j,k,\ell (dist)} M_{ij}M_{jk}M_{k\ell}M_{\ell i}, \qquad \text{where } M_{ij} = \widetilde{\Omega}_{ij} + W_{ij} + \delta_{ij} + \tilde{r}_{ij} + \epsilon_{ij},$$

and

$$Q_n^* = \sum_{i,j,k,\ell (dist)} M_{ij}^*M_{jk}^*M_{k\ell}^*M_{\ell i}^*, \qquad \text{where } M_{ij}^* \equiv \widetilde{\Omega}_{ij} + W_{ij} + \delta_{ij}.$$

We then introduce an intermediate variable:

$$\widetilde{Q}_n^* = \sum_{i,j,k,\ell(dist)} \widetilde{M}_{ij}^* \widetilde{M}_{jk}^* \widetilde{M}_{k\ell}^* \widetilde{M}_{\ell i}^*, \quad \text{where } \widetilde{M}_{ij}^* = \widetilde{\Omega}_{ij} + W_{ij} + \delta_{ij} + \tilde{r}_{ij}. \tag{66}$$

As a result, $(Q_n - Q_n^*)$ decomposes into

$$Q_n - Q_n^* = (\widetilde{Q}_n^* - Q_n^*) + (Q_n - \widetilde{Q}_n^*). \tag{67}$$

We note that Q_n can be expanded to the sum of $5^4 = 625$ post-expansion sums, each with the form

$$\sum_{i,j,k,\ell(dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i},$$

where each of $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ takes values in $\{\widetilde{\Omega}_{ij}, W_{ij}, \delta_{ij}, \tilde{r}_{ij}, \epsilon_{ij}\}$. Let $N_{\widetilde{\Omega}}$ be the (common) number of $\widetilde{\Omega}$ terms in each product and define $N_W, N_\delta, N_{\tilde{r}}, N_\epsilon$ similarly. Among the 625 post-expansion sums,

- $3^4 = 81$ of them are contained in Q_n^*,
- $4^4 - 3^4 = 175$ of them are contained in $(\widetilde{Q}_n^* - Q_n^*)$,
- and $5^4 - 4^4 = 369$ of them are contained in $(Q_n - \widetilde{Q}_n^*)$.

We shall study $(\widetilde{Q}_n^* - Q_n^*)$ and $(Q_n - \widetilde{Q}_n^*)$, separately.

In our analysis, one challenge is to deal with the random variable V that appears in the denominator in the expression of r_{ij}. The following lemma is useful and proved in Section G.4.

LEMMA G.10. *Suppose conditions of Theorem 4.3 hold. As $n \to \infty$, for any sequence x_n such that $\sqrt{\log(\|\theta\|_1)} \ll x_n \ll \|\theta\|_1$,*

$$\mathbb{E}\big[(\widetilde{Q}_n - Q_n)^2 \cdot I\{|V - v| > \|\theta\|_1 x_n\}\big] \to 0.$$

The next two lemmas are proved in Section G.4.

LEMMA G.11. *Suppose conditions of Theorem 4.3 hold. Write $\alpha = |\lambda_2|/\lambda_1$. As $n \to \infty$,*

- *Under the null hypothesis, $|\mathbb{E}[\widetilde{Q}_n^* - Q_n^*]| = o(\|\theta\|^4)$ and $\mathrm{Var}(\widetilde{Q}_n^* - Q_n^*) = o(\|\theta\|^8)$.*
- *Under the alternative hypothesis, $|\mathbb{E}[\widetilde{Q}_n^* - Q_n^*]| = o(\alpha^4\|\theta\|^8)$ and $\mathrm{Var}(\widetilde{Q}_n^* - Q_n^*) = o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6)$.*

LEMMA G.12. *Suppose conditions of Theorem 4.3 hold. Write $\alpha = |\lambda_2|/\lambda_1$. As $n \to \infty$,*

- *Under the null hypothesis, $|\mathbb{E}[Q_n - \widetilde{Q}_n^*]| = o(\|\theta\|^4)$ and $\mathrm{Var}(Q_n - \widetilde{Q}_n^*) = o(\|\theta\|^8)$.*
- *Under the alternative hypothesis, $|\mathbb{E}[Q_n - \widetilde{Q}_n^*]| = o(\alpha^4\|\theta\|^8)$ and $\mathrm{Var}(\widetilde{Q}_n^* - Q_n^*) = O(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6)$.*

Theorem 4.3 follows directly from (67) and Lemmas G.11-G.12.

G.4. Proof of Lemmas G.1-G.12.

G.4.1. *Proof of Lemma G.1.* Under the null hypothesis,

$$\widetilde{Q}_n = X_1 = \sum_{i,j,k,\ell (dist)} W_{ij} W_{jk} W_{k\ell} W_{\ell i}.$$

For mutually distinct indices (i,j,k,ℓ), $(W_{ij}, W_{jk}, W_{k\ell}, W_{\ell i})$ are independent of each other, each with mean zero. So $\mathbb{E}[W_{ij} W_{jk} W_{k\ell} W_{\ell i}] = 0$. It follows that

$$\mathbb{E}[\widetilde{Q}_n] = 0.$$

We now calculate the variance of $\widetilde{Q}_n$. Under the null hypothesis, $\Omega_{ij} = \theta_i \theta_j$; hence, $\mathrm{Var}(W_{ij}) = \Omega_{ij}(1 - \Omega_{ij}) = \theta_i\theta_j - \theta_i^2\theta_j^2 = \theta_i\theta_j[1 + O(\theta_{\max}^2)]$. It follows that

$$\begin{aligned} \mathrm{Var}(W_{ij} W_{jk} W_{k\ell} W_{\ell i}) &= \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 \cdot [1 + O(\theta_{\max}^2)]^4 \\ &= \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 \cdot [1 + O(\theta_{\max}^2)]. \end{aligned} \tag{68}$$

Note that each (i,j,k,ℓ) corresponds to a 4-cycle in a complete graph of n nodes. For (i,j,k,ℓ) and (i',j',k',ℓ'), we can write $W_{ij} W_{jk} W_{k\ell} W_{\ell i} \cdot W_{i'j'} W_{j'k'} W_{k'\ell'} W_{\ell' i'}$ in the form of $\prod_t (W_{i_t j_t})^{m_t}$, where $\{W_{i_t j_t}\}$ are mutually distinct with each other and m_t is the number of times that $W_{i_t j_t}$ appears in this product. If the two 4-cycles corresponding to (i,j,k,ℓ) and (i',j',k',ℓ') are not exactly overlapping, then at least two of m_t equals to 1. As a result, the mean of $\prod_t (W_{i_t j_t})^{m_t}$ is zero. In other words, we have argued that

(69) $\mathrm{Cov}(W_{ij} W_{jk} W_{k\ell} W_{\ell i},\ W_{i'j'} W_{j'k'} W_{k'\ell'} W_{\ell' i'}) = 0$ if the two cycles corresponding to (i,j,k,ℓ) and (i',j',k',ℓ') are not exactly overlapping.

In the sum over all distinct (i,j,k,ℓ), each 4-cycle is repeatedly counted by 8 times

$$\begin{gathered} (i,j,k,\ell), (j,k,\ell,i), (k,\ell,i,j), (\ell,i,j,k), \\ (\ell,k,j,i), (k,j,i,\ell), (j,i,\ell,k), (i,\ell,k,j). \end{gathered}$$

It follows that

$$\begin{aligned} \mathrm{Var}(\widetilde{Q}_n) &= \mathrm{Var}\Bigg(8 \sum_{\substack{\text{unique} \\ \text{4-cycles}}} W_{ij} W_{jk} W_{k\ell} W_{\ell i}\Bigg) \\ &= 64 \cdot \mathrm{Var}\Bigg(\sum_{\substack{\text{unique} \\ \text{4-cycles}}} W_{ij} W_{jk} W_{k\ell} W_{\ell i}\Bigg) \\ &= 64 \sum_{\substack{\text{unique} \\ \text{4-cycles}}} \mathrm{Var}\big(W_{ij} W_{jk} W_{k\ell} W_{\ell i}\big) \\ &= 8 \sum_{i,j,k,\ell (dist)} \mathrm{Var}\big(W_{ij} W_{jk} W_{k\ell} W_{\ell i}\big) \\ &= [1 + O(\theta_{\max}^2)] \cdot 8 \sum_{i,j,k,\ell (dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2, \end{aligned} \tag{70}$$

where the third line is from (69) and the last line is from (68). We then compute the right hand side of (70). Note that

$$\sum_{i,j,k,\ell (dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 = \|\theta\|^8 - \sum_{i,j,k,\ell (not\ dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2,$$

where

$$\sum_{i,j,k,\ell (not\ dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 \leq \binom{4}{2}\sum_{i,j,k}\theta_i^2\theta_j^2\theta_k^4 \leq C\|\theta\|^4\|\theta\|_4^4 = \|\theta\|^8 \cdot O\Big(\frac{\|\theta\|_4^4}{\|\theta\|^4}\Big).$$

Combining the above gives

$$\sum_{i,j,k,\ell (dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 = \|\theta\|^8 \cdot \Big[1 + O\Big(\frac{\|\theta\|_4^4}{\|\theta\|^4}\Big)\Big]. \tag{71}$$

We combine (70)-(71) and note that $\theta_{\max} = o(1)$ and $\|\theta\|_4^4/\|\theta\|^4 \leq (\|\theta\|^2\theta_{\max}^2)/\|\theta\|^4 = o(1)$. So,

$$\mathrm{Var}(\widetilde{Q}_n) = 8\|\theta\|^8 \cdot [1 + o(1)].$$

This completes the proof.

G.4.2. *Proof of Lemma G.2.* Under the null hypothesis,

$$\widetilde{Q}_n = X_1 = \sum_{i,j,k,\ell (dist)} W_{ij}W_{jk}W_{k\ell}W_{\ell i}.$$

In the proof of Theorem 3.2 of [3], it has been shown that $X_1/\sqrt{\mathrm{Var}(X_1)} \to N(0,1)$ in law (in the proof there, $X_1/\sqrt{\mathrm{Var}(X_1)}$ is denoted as $S_{n,n}$). Since $\mathbb{E}[X_1] = 0$, we can directly quote their results to get the desired claim.

G.4.3. *Proof of Lemma G.3.* We shall study the mean and variance of each of X_1-X_6 and then combine those results.

Consider X_1. We have analyzed this term under the null hypothesis. Under the alternative hypothesis, the difference is that we no longer have $\Omega_{ij} = \theta_i\theta_j$. Instead, we have an upper bound $\Omega_{ij} = \theta_i\theta_j(\pi_i' P\pi_j) \leq C\theta_i\theta_j$. Using similar proof as that for the null hypothesis, we can derive that

$$\mathbb{E}[X_1] = 0, \qquad \mathrm{Var}(X_1) \leq C\|\theta\|^8. \tag{72}$$

To get a lower bound for $\mathrm{Var}(X_1)$, we notice that $\mathrm{Var}(W_{ij}) = \Omega_{ij}(1-\Omega_{ij}) \geq \Omega_{ij}[1 - O(\theta_{\max}^2)] \geq \Omega_{ij}/2$; this inequality is true even when $\Omega_{ij} = 0$. It follows that

$$\mathrm{Var}(W_{ij}W_{jk}W_{k\ell}W_{\ell i}) \geq \frac{1}{16}\Omega_{ij}\Omega_{jk}\Omega_{k\ell}\Omega_{\ell i}.$$

Note that the second last line of (70) is still true. As a result,

$$\begin{aligned}
\mathrm{Var}(X_1) &= 8\sum_{i,j,k,\ell (dist)} \mathrm{Var}\big(W_{ij}W_{jk}W_{k\ell}W_{\ell i}\big) \\
&\geq \frac{1}{2}\sum_{i,j,k,\ell (dist)} \Omega_{ij}\Omega_{jk}\Omega_{k\ell}\Omega_{\ell i} \\
&= \frac{1}{2}\mathrm{tr}(\Omega^4) - \frac{1}{2}\sum_{i,j,k,\ell (not\ dist)} \Omega_{ij}\Omega_{jk}\Omega_{k\ell}\Omega_{\ell i} \\
&\geq \frac{1}{2}\mathrm{tr}(\Omega^4) - C\sum_{i,j,k,\ell (not\ dist)} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2 \\
&\geq \frac{1}{2}\mathrm{tr}(\Omega^4) - o(\|\theta\|^8),
\end{aligned}$$

where the last inequality is due to (71). Recall that $\lambda_1, \ldots, \lambda_K$ denote the K nonzero eigenvalues of Ω. By Lemma E.2, $\lambda_1 \geq C^{-1}\|\theta\|^2$. It follows that

$$\mathrm{tr}(\Omega^4) = \sum_{k=1}^{K} \lambda_k^4 \geq \lambda_1^4 \geq C^{-1}\|\theta\|^8.$$

Combining the above gives

$$\mathrm{Var}(X_1) \geq C^{-1}\|\theta\|^8. \tag{73}$$

So far, we have proved all claims about X_1.

Consider X_2. Recall that

$$X_2 = \sum_{i,j,k,\ell (dist)} \widetilde{\Omega}_{ij} W_{jk} W_{k\ell} W_{\ell i}.$$

It is easy to see that $\mathbb{E}[X_2] = 0$. Below, we bound its variance. Each index choice (i,j,k,ℓ) defines a undirected path j-k-ℓ-i in the complete graph of n nodes. If the two paths j-k-ℓ-i and j'-k'-ℓ'-i' are not exactly overlapping, then $W_{jk}W_{k\ell}W_{\ell i} \cdot W_{j'k'}W_{k'\ell'}W_{\ell' i'}$ have mean zero. In the sum above, each unique path j-k-ℓ-i is counted twice as (i,j,k,ℓ) and (j,i,ℓ,k). Mimicking the argument in (70), we immediately have

$$\begin{aligned}\mathrm{Var}(X_2) &= 2 \sum_{i,j,k,\ell (dist)} \mathrm{Var}\big(\widetilde{\Omega}_{ij} W_{jk} W_{k\ell} W_{\ell i}\big) \\ &= 2 \sum_{i,j,k,\ell (dist)} \widetilde{\Omega}_{ij}^2 \cdot \mathrm{Var}\big(W_{jk} W_{k\ell} W_{\ell i}\big).\end{aligned}$$

By Lemma E.5, $|\widetilde{\Omega}_{ij}| \leq |\lambda_2| \|\theta\|^{-2}\theta_i\theta_j$. In our notations, $\alpha = |\lambda_2|/\lambda_1$; additionally, by Lemma E.2, $\lambda_1 \leq C\|\theta\|^2$. Combining them gives

$$|\widetilde{\Omega}_{ij}| \leq C\alpha\theta_i\theta_j. \tag{74}$$

Moreover, $\mathrm{Var}(W_{jk}W_{k\ell}W_{\ell i}) \leq \Omega_{jk}\Omega_{k\ell}\Omega_{\ell i} \leq C\theta_j\theta_k^2\theta_\ell^2\theta_i$. It follows that

$$\begin{aligned}\mathrm{Var}(X_2) &\leq C \sum_{i,j,k,\ell (dist)} (\alpha\theta_i\theta_j)^2 \cdot \theta_j\theta_k^2\theta_\ell^2\theta_i \\ &\leq C\alpha^2 \sum_{i,j,k,\ell} \theta_i^3\theta_j^3\theta_k^2\theta_\ell^2 \\ &\leq C\alpha^2\|\theta\|^4\|\theta\|_3^6.\end{aligned}$$

Since $\|\theta\|_3^3 \leq \theta_{\max}\sum_i \theta_i^2 = \theta_{\max}\|\theta\|^2$, the right hand side is $\leq C\alpha^2\|\theta\|^8\theta_{\max}^2$. Note that $\alpha \leq 1$ and $\theta_{\max} \to 0$. So, this term is $o(\|\theta\|^8)$. We have proved all claims about X_2.

Consider X_3. Recall that

$$X_3 = \sum_{i,j,k,\ell (dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk} W_{k\ell} W_{\ell i} = \sum_{i,k,\ell (dist)} \Big(\sum_{j \notin \{i,k,\ell\}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk} \Big) W_{k\ell} W_{\ell i}.$$

It is easy to see that $\mathbb{E}[X_3] = 0$. We then study its variance. We note that for $W_{k\ell}W_{\ell i}$ and $W_{k'\ell'}W_{\ell' i'}$ to be correlated, we must have that $(k',\ell',i') = (k,\ell,i)$ or $(k',\ell',i') = (i,\ell,k)$; in other words, the two underlying paths k-ℓ-i and k'-ℓ'-i' have to be equal. Mimicking the argument in (70), we have

$$\begin{aligned}\mathrm{Var}(X_3) &\le C \sum_{i,k,\ell(dist)} \mathrm{Var}\Big[\Big(\sum_{j\notin\{i,k,\ell\}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\Big) W_{k\ell}W_{\ell i}\Big] \\ &\le C \sum_{i,k,\ell(dist)} \Big(\sum_{j\notin\{i,k,\ell\}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\Big)^2 \cdot \mathrm{Var}(W_{k\ell}W_{\ell i}).\end{aligned}$$

By (74),

$$\Big|\sum_{j\notin\{i,k,\ell\}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\Big| \le C\sum_j \alpha^2\theta_i\theta_j^2\theta_k \le C\alpha^2\|\theta\|^2\cdot\theta_i\theta_k.$$

Combining the above gives

$$\begin{aligned}\mathrm{Var}(X_3) &\le C\sum_{i,k,\ell} (\alpha^2\|\theta\|^2\theta_i\theta_k)^2\cdot\theta_k\theta_\ell^2\theta_i \\ &\le C\alpha^4\|\theta\|^4\sum_{i,k,\ell}\theta_i^3\theta_k^3\theta_\ell^2 \\ &\le C\alpha^4\|\theta\|^6\|\theta\|_3^6.\end{aligned}$$

Since $\|\theta\|\to\infty$, the right hand side is $o(\alpha^4\|\theta\|^8\|\theta\|_3^6)$. We have proved all claims about X_3.

Consider X_4. Recall that

$$X_4 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}W_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{k\ell}W_{jk}W_{\ell i}.$$

It is easy to see that $\mathbb{E}[X_4]=0$. To calculate its variance, note that $W_{jk}W_{\ell i}$ and $W_{j'k'}W_{\ell' i'}$ are uncorrelated unless (i) $\{j',k'\}=\{j,k\}$ and $\{\ell',i'\}=\{\ell,i\}$ or (ii) $\{j',k'\}=\{\ell,i\}$ and $\{\ell',i'\}=\{j,k\}$. Mimicking the argument in (70), we immediately have

$$\begin{aligned}\mathrm{Var}(X_4) &\le C\sum_{i,j,k,\ell(dist)} \mathrm{Var}\big(\widetilde{\Omega}_{ij}\widetilde{\Omega}_{k\ell}W_{jk}W_{\ell i}\big) \\ &\le C\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}^2\widetilde{\Omega}_{k\ell}^2\cdot\mathrm{Var}(W_{jk}W_{\ell i}) \\ &\le C\sum_{i,j,k,\ell}(\alpha\theta_i\theta_j)^2(\alpha\theta_k\theta_\ell)^2\cdot\theta_j\theta_k\theta_\ell\theta_i \\ &\le C\alpha^4\sum_{i,j,k,\ell}\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3 \\ &\le C\alpha^4\|\theta\|_3^{12}.\end{aligned}$$

Since $\|\theta\|_3^3\le\theta_{\max}\|\theta\|^2=o(\|\theta\|^2)$, the right hand side is $o(\|\theta\|^8)$. This proves the claims of X_4.

Consider X_5. Recall that

$$X_5 = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} = 2\sum_{i<\ell}\Big(\sum_{\substack{j,k\notin\{i,\ell\}\\ j\neq k}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\Big)W_{\ell i}.$$

It is easily seen that $\mathbb{E}[X_5]=0$. Furthermore, we have

$$\mathrm{Var}(X_5) = 2\sum_{i<\ell}\Big(\sum_{\substack{j,k\notin\{i,\ell\}\\ j\neq k}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\Big)^2\cdot\mathrm{Var}(W_{\ell i}). \tag{75}$$

By (74),

$$\Big|\sum_{\substack{j,k\notin\{i,\ell\}\\ j\neq k}} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\Big| \leq C\sum_{j,k}\alpha^3\theta_i\theta_j^2\theta_k^2\theta_\ell \leq C\alpha^3\|\theta\|^4\cdot\theta_i\theta_\ell$$

We plug it into (75) and use $\mathrm{Var}(W_{\ell i}) \leq \Omega_{\ell i} \leq C\theta_\ell\theta_i$. It yields that

$$\begin{aligned}\mathrm{Var}(X_5) &\leq C\sum_{\ell,i(dist)}(\alpha^3\|\theta\|^4\theta_i\theta_\ell)^2\cdot\theta_\ell\theta_i\\ &\leq C\alpha^6\|\theta\|^8\sum_{\ell,i}\theta_i^3\theta_\ell^3\\ &\leq C\alpha^6\|\theta\|^8\|\theta\|_3^6. \qquad (76)\end{aligned}$$

This proves the claims of X_5.

Consider X_6. Recall that

$$X_6 = \sum_{i,j,k,\ell(dist)}\widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} = \mathrm{tr}(\widetilde{\Omega}^4) - \sum_{i,j,k,\ell(not\,dist)}\widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}.$$

This is a non-stochastic number, so its variance is zero and its mean is X_6 itself. By Lemma E.5, $|\lambda_2| \leq \|\widetilde{\Omega}\| \leq C|\lambda_2|$. Since $\|\widetilde{\Omega}\|^4 \leq \mathrm{tr}(\widetilde{\Omega}^4) \leq K\|\widetilde{\Omega}\|^4$, we immediately have $\mathrm{tr}(\widetilde{\Omega}^4) \asymp \|\widetilde{\Omega}\|^4 \asymp |\lambda_2|^4$. Additionally, $|\lambda_2| = \alpha\lambda_1$ in our notation, and $\lambda_1 \asymp \|\theta\|^2$ by Lemma E.2. It follows that

$$\mathrm{tr}(\widetilde{\Omega}^4) \asymp |\lambda_2|^4 \asymp \alpha^4\|\theta\|^8.$$

At the same time, by (74), $|\widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}| \leq C\alpha^4\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2$. We thus have

$$\begin{aligned}|X_6 - \mathrm{tr}(\widetilde{\Omega}^4)| &\leq C\alpha^4\sum_{i,j,k,\ell(not\,dist)}\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\\ &\leq C\alpha^4\sum_{i,j,k}\theta_i^2\theta_j^2\theta_k^4\\ &\leq C\alpha^4\|\theta\|^4\|\theta\|_4^4 = o(\alpha^4\|\theta\|^8),\end{aligned}$$

where the last equality is due to $\|\theta\|_4^4 \leq \theta_{\max}^2\|\theta\|^2 = o(\|\theta\|^4)$. Combining the above gives

$$X_6 = \mathrm{tr}(\widetilde{\Omega}^4)\cdot[1+o(1)].$$

This proves the claims of X_6.

Last, we combine the results for X_1-X_6 to study $\widetilde{Q}_n$. Note that

$$\widetilde{Q}_n = X_1 + 4X_2 + 4X_3 + 2X_4 + 4X_5 + X_6.$$

Only X_6 has a nonzero mean. So,

$$\mathbb{E}[\widetilde{Q}_n] = \mathbb{E}[X_6] = \mathrm{tr}(\widetilde{\Omega}^4)\cdot[1+o(1)].$$

At the same time, given random variables $Z_1, Z_2, \ldots, Z_m$, $\mathrm{Var}(\sum_{k=1}^m Z_k) = \sum_k \mathrm{Var}(Z_k) + \sum_{k\neq\ell}\mathrm{Cov}(Z_k, Z_\ell) \leq \sum_k \mathrm{Var}(Z_k) + \sum_{k\neq\ell}\sqrt{\mathrm{Var}(Z_k)\mathrm{Var}(Z_\ell)} \leq m^2\max_k\{\mathrm{Var}(Z_k)\}$. We thus have

$$\mathrm{Var}(\widetilde{Q}_n) \leq C\max_{1\leq k\leq 6}\mathrm{Var}(X_k) \leq C\big(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6\big).$$

The proof of this lemma is now complete.

G.4.4. *Proof of Lemma G.4.* Recall that $U_a = 4Y_1 = 4\sum_{i,j,k,\ell(dist)} \delta_{ij} W_{jk}W_{k\ell}W_{\ell i}$. By definition, $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$. It follows that

$$U_a = 4 \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}W_{k\ell}W_{\ell i} + 4 \sum_{i,j,k,\ell(dist)} \eta_j(\eta_i - \tilde{\eta}_i)W_{jk}W_{k\ell}W_{\ell i}.$$

In the second sum, if we relabel $(i,j,k,\ell) = (j', i', \ell', k')$, it becomes

$$4 \sum_{i',j',k',\ell'(dist)} \eta_{i'}(\eta_{j'} - \tilde{\eta}_{j'})W_{i'\ell'}W_{\ell' k'}W_{k' j'} = 4 \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{i\ell}W_{\ell k}W_{kj},$$

which is the same as the first term. It follows that

$$U_a = 8 \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}W_{k\ell}W_{\ell i}.$$

By definition, $\eta_j = \frac{1}{\sqrt{v}}\sum_{s\neq j}\mathbb{E}A_{js}$ and $\tilde{\eta}_j = \frac{1}{\sqrt{v}}\sum_{s\neq j} A_{js}$. Hence,

$$\tilde{\eta}_j - \eta_j = \frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}. \tag{77}$$

We then re-write

$$\begin{aligned} U_a &= -8 \sum_{i,j,k,\ell(dist)} \eta_i\Big(\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)W_{jk}W_{k\ell}W_{\ell i} \\ &= -\frac{8}{\sqrt{v}} \sum_{\substack{i,j,k,\ell(dist)\\ s\neq j}} \eta_i W_{js}W_{jk}W_{k\ell}W_{\ell i}. \end{aligned}$$

In the summand, (i,j,k,ℓ) are distinct, but s is only required to be distinct from j. We consider two different cases: (a) the case of $s = k$, where the summand becomes $W_{jk}^2 W_{k\ell}W_{\ell i}$, and (b) the case of $s \neq k$. Correspondingly, we write

$$\begin{aligned} U_a &= -\frac{8}{\sqrt{v}} \sum_{i,j,k,\ell(dist)} \eta_i W_{jk}^2 W_{k\ell}W_{\ell i} - \frac{8}{\sqrt{v}} \sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{j,k\}}} \eta_i W_{js}W_{jk}W_{k\ell}W_{\ell i} \\ &\equiv U_{a1} + U_{a2}. \end{aligned} \tag{78}$$

It is easy to see that the summands in both sums have mean zero. Therefore,

$$\mathbb{E}[U_a] = 0.$$

Next, we bound the variance of U_a. Since $\mathrm{Var}(U_a) \leq 2\mathrm{Var}(U_{a1}) + 2\mathrm{Var}(U_{a2})$, it suffices to bound the variances of U_{a1} and U_{a2}. Consider U_{a1}. Note that

$$\mathrm{Var}(U_{a1}) = \frac{64}{v} \sum_{\substack{i,j,k,\ell(dist)\\ i',j',k',\ell'(dist)}} \eta_i\eta_{i'} \cdot \mathbb{E}[W_{jk}^2 W_{k\ell}W_{\ell i}W_{j'k'}^2 W_{k'\ell'}W_{\ell' i'}]. \tag{79}$$

By definition, $v = 1_n'(\mathbb{E}A)1_n = 1_n'\Omega 1_n - \sum_i \Omega_{ii}$. Since $\Omega_{ii} \leq \theta_i^2$, it implies $v = 1_n'\Omega 1_n - O(\|\theta\|^2) = 1_n'\Omega 1_n + o(\|\theta\|_1^2)$. Moreover, we note that $1_n'\Omega 1_n \leq C\sum_{i,j}\theta_i\theta_j \leq C\|\theta\|_1^2$, and by Lemma E.4, $1_n'\Omega 1_n \geq C^{-1}\|\theta\|_1^2$. Combining these results gives

$$C^{-1}\|\theta\|_1^2 \leq v \leq C\|\theta\|_1^2. \tag{80}$$

Moreover, $\eta_i = \frac{1}{\sqrt{v}}\sum_{s\neq i}\Omega_{is} \leq \frac{C}{\|\theta\|_1}\sum_s \theta_i\theta_s$. This gives

$$0 \leq \eta_i \leq C\theta_i, \qquad \text{for all } 1 \leq i \leq n. \tag{81}$$

We plug (80)-(81) into (79) and find out that

$$\mathrm{Var}(U_{a1}) \le \frac{C}{\|\theta\|_1^2} \sum_{\substack{i,j,k,\ell (dist)\\ i',j',k',\ell' (dist)}} \theta_i\theta_{i'} \cdot \mathbb{E}[W_{jk}^2 W_{k\ell} W_{\ell i} W_{j'k'}^2 W_{k'\ell'} W_{\ell' i'}].$$

In order for the summand to be nonzero, all W terms have to be perfectly paired. By elementary calculations,

$$\theta_i\theta_{i'}\mathbb{E}[W_{jk}^2 W_{k\ell} W_{\ell i} W_{j'k'}^2 W_{k'\ell'} W_{\ell' i'}] = \begin{cases} \theta_i^2\, \mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j'k}^2], & \text{if } (\ell',k',i')\text{=}(\ell,k,i); \\ \theta_i\theta_k\, \mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j'i}^2], & \text{if } (\ell',k',i')\text{=}(\ell,i,k); \\ \theta_i\theta_j\, \mathbb{E}[W_{jk}^3 W_{k\ell}^2 W_{\ell i}^3], & \text{if } (j',k')\text{=}(i,\ell),\ (i',\ell')\text{=}(j,k); \\ 0, & \text{otherwise.} \end{cases}$$

Here, (i,j,k,ℓ) are distinct. In the second case above, $(W_{jk}^2, W_{k\ell}^2, W_{\ell i}^2, W_{j'i}^2)$ are independent of each other, no matter $j=j'$ or $j\neq j'$ (we remark that $j'\neq \ell$, because $j'\notin\{i',k',\ell'\} = \{i,k,\ell\}$). It follows that $\mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j'i}^2] \le \Omega_{jk}\Omega_{k\ell}\Omega_{\ell i}\Omega_{j'i} \le C\theta_i^2\theta_j\theta_k^2\theta_\ell^2\theta_{j'}$. In the first case, when $j\neq j'$, $\mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j'k}^2] \le \Omega_{jk}\Omega_{k\ell}\Omega_{\ell i}\Omega_{j'k} \le C\theta_i\theta_j\theta_k^3\theta_\ell^2\theta_{j'}$; when $j=j'$, it holds that $\mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j'k}^2] = \mathbb{E}[W_{jk}^4 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i\theta_j\theta_k^2\theta_\ell^2$. In the third case, $(W_{jk}^3, W_{k\ell}^2, W_{\ell i}^3)$ are mutually independent, so $\mathbb{E}[W_{jk}^2 W_{k\ell}^2 W_{\ell i}^2] \le \Omega_{jk}\Omega_{k\ell}\Omega_{\ell i} \le C\theta_i\theta_j\theta_k^2\theta_\ell^2$. We then have

$$\theta_i\theta_{i'}\mathbb{E}[W_{jk}^2 W_{k\ell} W_{\ell i} W_{j'k'}^2 W_{k'\ell'} W_{\ell' i'}] \le \begin{cases} C\theta_i^3\theta_j\theta_k^2\theta_\ell^2, & \text{if } (\ell',k',i') = (\ell,k,i),\ j'=j; \\ C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_{j'}, & \text{if } (\ell',k',i') = (\ell,k,i),\ j'\neq j; \\ C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_{j'}, & \text{if } (\ell',k',i') = (\ell,i,k); \\ C\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2, & \text{if } (j',k')\text{=}(i,\ell),\ (i',\ell')\text{=}(j,k); \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} \mathrm{Var}(U_{a1}) &\le \frac{C}{\|\theta\|_1^2}\Big(\sum_{i,j,k,\ell}\theta_i^3\theta_j\theta_k^2\theta_\ell^2 + \sum_{i,j,k,\ell,j'}\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_{j'} + \sum_{i,j,k,\ell}\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\Big) \\ &\le \frac{C}{\|\theta\|_1^2}\big(\|\theta\|^4\|\theta\|_3^3\|\theta\|_1 + \|\theta\|^2\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^8\big) \\ &\le C\|\theta\|^2\|\theta\|_3^6, \end{aligned} \tag{82}$$

where we obtain the last inequality as follows: By Cauchy-Schwarz inequality, $\|\theta\|^4 = (\sum_i \theta_i^{1/2}\cdot\theta^{3/2})^2 \le (\sum_i\theta_i)(\sum_i\theta_i^3) \le \|\theta\|_1\|\theta\|_3^3$; therefore, $\|\theta\|^8 \le \|\theta\|^4\|\theta\|_3^3\|\theta\|_1 \le \|\theta\|_3^6\|\theta\|_1^2$. We then consider U_{a2}. Define

$$\mathcal{P}_5^* = \left\{ \begin{matrix}\text{path } i\text{-}\ell\text{-}k\text{-}j\text{-}s \text{ in a complete} \\ \text{graph with } n \text{ nodes}\end{matrix} : \begin{matrix}\text{nodes } i,j,k,\ell \text{ are distinct,} \\ \text{and node } s \text{ is different from } j,\, k\end{matrix} \right\}.$$

Fix a path i-ℓ-k-j-s in $\mathcal{P}_5^*$. If $s\notin\{i,\ell\}$, then this path is counted twice in the definition of U_{a2}, as i-ℓ-k-j-s and s-j-k-ℓ-i, respectively. If $s\in\{i,\ell\}$, then it is counted only once in the definition of U_{a2}. Hence, we can re-write

$$U_{a2} = -\frac{8}{\sqrt{v}}\sum_{\substack{\text{path in } \mathcal{P}_5^* \\ s\notin\{i,\ell\}}} (\eta_i+\eta_s) W_{sj}W_{jk}W_{k\ell}W_{\ell i} - \frac{8}{\sqrt{v}}\sum_{\substack{\text{path in } \mathcal{P}_5^* \\ s\in\{i,\ell\}}} \eta_i W_{sj}W_{jk}W_{k\ell}W_{\ell i}.$$

For two distinct paths in $\mathcal{P}_5^*$, the corresponding summands are uncorrelated with each other. It follows that

$$\begin{aligned}
\mathrm{Var}(U_{a2}) &= \frac{64}{v}\sum_{\substack{\text{path in } \mathcal{P}_5^* \\ s\notin\{i,\ell\}}}(\eta_i+\eta_s)^2\,\mathrm{Var}(W_{sj}W_{jk}W_{k\ell}W_{\ell i}) \\
&\quad + \frac{64}{v}\sum_{\substack{\text{path in } \mathcal{P}_5^* \\ s\in\{i,\ell\}}}\eta_i^2\,\mathrm{Var}(W_{sj}W_{jk}W_{k\ell}W_{\ell i}) \\
&\le \frac{C}{v}\sum_{i,j,k,\ell,s}(\eta_i^2+\eta_s^2)\cdot\theta_i\theta_j^2\theta_k^2\theta_\ell^2\theta_s \\
&\le \frac{C}{\|\theta\|_1^2}\sum_{i,j,k,\ell,s}(\theta_i^3\theta_j^2\theta_k^2\theta_\ell^2\theta_s+\theta_i\theta_j^2\theta_k^2\theta_\ell^2\theta_s^3) \\
&\le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}. \qquad (83)
\end{aligned}$$

By Cauchy-Schwarz inequality, $\|\theta\|^4\le\|\theta\|_1\|\theta\|_3^3$, so the right hand side of (83) is $\le C\|\theta\|^2\|\theta\|_3^6$. Combining it with (82) gives

$$\mathrm{Var}(U_a)\le C\|\theta\|^2\|\theta\|_3^6=o(\|\theta\|^8).$$

This proves the claim.

G.4.5. *Proof of Lemma G.5.* It suffices to prove the claims for each of Y_1-Y_6. Consider Y_1. We have analyzed this term under the null hypothesis. Using similar proof, we can easily derive that

$$\mathbb{E}[Y_1]=0,\qquad \mathrm{Var}(Y_1)\le C\|\theta\|^2\|\theta\|_3^6=o(\|\theta\|^8).$$

Consider Y_2. Using the definition of Y_2 and the expression of $\tilde{\eta}_i$ in (77), we have

$$\begin{aligned}
Y_2 &= \sum_{i,j,k,\ell(dist)}\delta_{ij}\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i} \\
&= \sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde{\eta}_j)\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i}+\sum_{i,j,k,\ell(dist)}\eta_j(\eta_i-\tilde{\eta}_i)\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i} \\
&= \frac{1}{\sqrt{v}}\sum_{i,j,k,\ell(dist)}\eta_i\Big(-\sum_{s\neq j}W_{js}\Big)\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i}+\frac{1}{\sqrt{v}}\sum_{i,j,k,\ell(dist)}\eta_j\Big(-\sum_{s\neq i}W_{is}\Big)\widetilde{\Omega}_{jk}W_{k\ell}W_{\ell i} \\
&= -\frac{1}{\sqrt{v}}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq j}}\eta_i\widetilde{\Omega}_{jk}W_{js}W_{k\ell}W_{\ell i}-\frac{1}{\sqrt{v}}\sum_{\substack{i,k,\ell(dist)\\ s\neq i}}\Big(\sum_{j\notin\{i,k,\ell\}}\eta_j\widetilde{\Omega}_{jk}\Big)W_{is}W_{k\ell}W_{\ell i}.
\end{aligned}$$

In the second sum above, we further separate two cases, $s=\ell$ and $s\neq\ell$. It then gives rise to three terms:

$$\begin{aligned}
Y_2 = &-\frac{1}{\sqrt{v}}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq j}}\eta_i\widetilde{\Omega}_{jk}W_{js}W_{k\ell}W_{\ell i} \\
&-\frac{1}{\sqrt{v}}\sum_{i,k,\ell(dist)}\Big(\sum_{j\notin\{i,k,\ell\}}\eta_j\widetilde{\Omega}_{jk}\Big)W_{i\ell}^2W_{k\ell}
\end{aligned}$$

$$-\frac{1}{\sqrt{v}}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\}}}\Big(\sum_{j\notin\{i,k,\ell\}}\eta_j\widetilde{\Omega}_{jk}\Big)W_{is}W_{k\ell}W_{\ell i}$$

$$\equiv Y_{2a}+Y_{2b}+Y_{2c}. \tag{84}$$

Since (i,j,k,ℓ) are distinct, it is easy to see that all three terms have mean zero. We thus have

$$\mathbb{E}[Y_2]=0.$$

Below, we calculate the variances. First, we bound the variance of Y_{2a}. Each (i,j,k,ℓ,s) is associated with a length-3 path i-k-ℓ and an edge j-s in the complete graph. For (i,j,k,ℓ,s) and (i',j',k',ℓ',s'), if the associated path and edge are the same, then we group them together. Given a length-3 path i-k-ℓ and an edge j-s (such that the edge is not in the path), they are counted four times in the definition of Y_{2a}, as (i) i-k-ℓ and j-s, (ii) i-k-ℓ and s-j, (iii) ℓ-k-i and j-s, (iv) ℓ-k-i and s-j, so we group these four summands together. After grouping the summands, we re-write

$$Y_{2a}=-\frac{1}{\sqrt{v}}\sum_{\substack{\text{length-3}\\ \text{path}}}\sum_{\substack{\text{edge not}\\ \text{in the path}}}\big(\eta_i\widetilde{\Omega}_{jk}+\eta_i\widetilde{\Omega}_{sk}+\eta_k\widetilde{\Omega}_{ji}+\eta_k\widetilde{\Omega}_{si}\big)W_{js}W_{k\ell}W_{\ell i}.$$

In this new expression of Y_{2a}, two summands are correlated only when the underlying path&edge pairs are exactly the same. Additionally, by (74) and (81),

$$\big|\eta_i\widetilde{\Omega}_{jk}+\eta_i\widetilde{\Omega}_{sk}+\eta_k\widetilde{\Omega}_{ji}+\eta_k\widetilde{\Omega}_{si}\big|\le C\alpha(\theta_j+\theta_s)\theta_i\theta_k.$$

It follows that

$$\begin{aligned}
\mathrm{Var}(Y_{2a}) &\le \frac{C}{v}\sum_{i,j,k,\ell,s}\alpha^2(\theta_j+\theta_s)^2\theta_i^2\theta_k^2\cdot\mathrm{Var}(W_{js}W_{k\ell}W_{\ell i})\\
&\le \frac{C}{\|\theta\|_1^2}\sum_{i,j,k,\ell,s}\alpha^2(\theta_j+\theta_s)^2\theta_i^2\theta_k^2\cdot\theta_i\theta_j\theta_k\theta_\ell^2\theta_s\\
&\le \frac{C\alpha^2}{\|\theta\|_1^2}\sum_{i,j,k,\ell,s}(\theta_i^3\theta_j^3\theta_k^3\theta_\ell^2\theta_s+\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_s^3)\\
&\le \frac{C\alpha^2\|\theta\|^2\|\theta\|_3^9}{\|\theta\|_1}.
\end{aligned}\tag{85}$$

Second, we bound the variance of Y_{2b}. Write $\beta_{ik\ell}=\sum_{j\notin\{i,k,\ell\}}\eta_j\widetilde{\Omega}_{jk}$. By (74) and (81), $|\beta_{ik\ell}|\le C\sum_j\theta_j\cdot\alpha\theta_j\theta_k\le C\alpha\|\theta\|^2\theta_k$. Using this notation,

$$Y_{2b}=\frac{1}{v}\sum_{i,j,k,\ell(dist)}\beta_{ik\ell}W_{i\ell}^2W_{k\ell},\qquad\text{where}\quad |\beta_{ik\ell}|\le C\alpha\|\theta\|^2\theta_k.$$

It follows that

$$\begin{aligned}
\mathrm{Var}(Y_{2b})=\mathbb{E}[Y_{2b}^2] &\le \frac{C}{v}\sum_{\substack{i,k,\ell(dist)\\ i',k',\ell'(dist)}}\beta_{ik\ell}\beta_{i'k'\ell'}\cdot\mathbb{E}[W_{i\ell}^2W_{k\ell}W_{i'\ell'}^2W_{k'\ell'}]\\
&\le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^2}\sum_{\substack{i,k,\ell(dist)\\ i',k',\ell'(dist)}}\theta_k\theta_{k'}\cdot\mathbb{E}[W_{i\ell}^2W_{k\ell}W_{i'\ell'}^2W_{k'\ell'}].
\end{aligned}$$

The summand is nonzero only when the two variables $W_{k\ell}$ and $W_{k'\ell'}$ equal to each other or when each of them equals to some other squared variables. By elementary calculations,

$$\begin{aligned} &\theta_k\theta_{k'}\cdot\mathbb{E}[W_{i\ell}^2W_{k\ell}W_{i'\ell'}^2W_{k'\ell'}] \\ &= \begin{cases} \theta_k^2\,\mathbb{E}[W_{i\ell}^4W_{k\ell}^2]\leq C\theta_i\theta_k^3\theta_\ell^2, & \text{if } (k',\ell')=(k,\ell),\ i'=i; \\ \theta_k^2\,\mathbb{E}[W_{i\ell}^2W_{k\ell}^2W_{i'\ell}^2]\leq C\theta_i\theta_k^3\theta_\ell^3\theta_{i'}, & \text{if } (k',\ell')=(k,\ell),\ i'\neq i; \\ \theta_k\theta_\ell\,\mathbb{E}[W_{i\ell}^2W_{k\ell}^2W_{i'k}^2]\leq C\theta_i\theta_k^3\theta_\ell^3\theta_{i'}, & \text{if } (k',\ell')=(\ell,k); \\ \theta_k^2\,\mathbb{E}[W_{i\ell}^3W_{k\ell}^3]\leq C\theta_i\theta_k^3\theta_\ell^2, & \text{if } \ell'=\ell,\ (i',k')=(i,k); \\ \theta_k\theta_i\,\mathbb{E}[W_{i\ell}^3W_{k\ell}^3]\leq C\theta_i^2\theta_k^2\theta_\ell^2, & \text{if } \ell'=\ell,\ (i',k')=(k,i); \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

As a result,

$$\begin{aligned} \mathrm{Var}(Y_{2b}) &\leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^2}\Big(\sum_{i,k,\ell}\theta_i\theta_k^3\theta_\ell^2+\sum_{i,k,\ell,i'}\theta_i\theta_k^3\theta_\ell^3\theta_{i'}+\sum_{i,k,\ell}\theta_i^2\theta_k^2\theta_\ell^2\Big) \\ &\leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^2}\big(\|\theta\|_3^3\|\theta\|^2\|\theta\|_1+\|\theta\|_3^6\|\theta\|_1^2+\|\theta\|^6\big) \\ &\leq C\alpha^2\|\theta\|^4\|\theta\|_3^6, \end{aligned} \tag{86}$$

where to get the last inequality we have used $\|\theta\|^6\ll\|\theta\|^8\leq(\|\theta\|_1\|\theta\|_3^3)^2$ and $\|\theta\|_3^3\|\theta\|^2\|\theta\|_1\ll\|\theta\|_3^3\|\theta\|^4\|\theta\|_1\leq(\|\theta\|_1\|\theta\|_3^3)^2$. Last, we bound the variance of Y_{2c}. Let $\beta_{ik\ell}=\sum_{j\notin\{i,k,\ell\}}\eta_j\widetilde{\Omega}_{jk}$ be the same as above. We write

$$Y_{2c}=\frac{1}{\sqrt{v}}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\}}}\beta_{ik\ell}W_{is}W_{k\ell}W_{\ell i},\qquad \text{where}\quad |\beta_{ik\ell}|\leq C\alpha\|\theta\|^2\theta_k.$$

For $\mathbb{E}[W_{is}W_{k\ell}W_{\ell i}\cdot W_{i's'}W_{k'\ell'}W_{\ell'i'}]$ to be nonzero, it has to be the case that $(W_{is},W_{k\ell},W_{\ell i})$ and $(W_{i's'},W_{k'\ell'},W_{\ell'i'})$ are the same set of variables, up to an order permutation. For each fixed (i,k,ℓ,s), there are only a constant number of (i',k',ℓ',s') such that the above is satisfied. As we have argued many times before (e.g., see (70)), it is true that

$$\begin{aligned} \mathrm{Var}(Y_{2c}) &\leq \frac{C}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\}}}\beta_{ik\ell}^2\cdot\mathrm{Var}(W_{is}W_{k\ell}W_{\ell i}) \\ &\leq \frac{C}{\|\theta\|_1^2}\sum_{i,k,\ell,s}(\alpha\|\theta\|^2\theta_k)^2\cdot\theta_i^2\theta_k\theta_\ell^2\theta_s \\ &\leq \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1}. \end{aligned} \tag{87}$$

We now combine the variances of Y_{2a}-Y_{2c}. Since $\|\theta\|_3^3\leq\theta_{\max}^2\|\theta\|_1\ll\|\theta\|_1$, the right hand side is (85) is $o(\alpha^2\|\theta\|^2\|\theta\|_3^6)=o(\alpha^2\|\theta\|^4\|\theta\|_3^6)$. Since $\|\theta\|^4\leq\|\theta\|_1\|\theta\|_3^3$, the right hand side is (87) is $\leq C\alpha^2\|\theta\|^4\|\theta\|_3^6$. It follows that

$$\mathrm{Var}(Y_2)\leq C\alpha^2\|\theta\|^4\|\theta\|_3^6=o(\|\theta\|^8).$$

This proves the claims of Y_2.

Consider Y_3. By definition,

$$Y_3=\sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde{\eta}_j)W_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i}+\sum_{i,j,k,\ell(dist)}\eta_j(\eta_i-\tilde{\eta}_i)W_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i}.$$

In the second sum, if we relabel $(i,j,k,\ell)=(j',i',\ell',k')$, it can be written as $\sum_{i',j',k',\ell'(dist)}\eta_{i'}(\eta_{j'}-\tilde\eta_{j'})W_{i'\ell'}\widetilde\Omega_{\ell'k'}W_{k'j'}$. This shows that the second sum is indeed equal to the first sum. As a result,

$$\begin{aligned}
Y_3 &= 2\sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde\eta_j)W_{jk}\widetilde\Omega_{k\ell}W_{\ell i}\\
&= 2\sum_{i,j,k,\ell(dist)}\eta_i\Big(-\frac{1}{\sqrt v}\sum_{s\neq j}W_{js}\Big)W_{jk}\widetilde\Omega_{k\ell}W_{\ell i}\\
&= -\frac{2}{\sqrt v}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq j}}\eta_i\widetilde\Omega_{k\ell}W_{js}W_{jk}W_{\ell i}\\
&= -\frac{2}{\sqrt v}\sum_{i,j,k,\ell(dist)}\eta_i\widetilde\Omega_{k\ell}W_{jk}^2W_{\ell i}-\frac{2}{\sqrt v}\sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{j,k\}}}\eta_i\widetilde\Omega_{k\ell}W_{js}W_{jk}W_{\ell i}\\
&\equiv Y_{3a}+Y_{3b}, \qquad (88)
\end{aligned}$$

where the second line is from (77) and the second last line is from dividing all summands into two cases of $s=k$ and $s\neq k$. Both terms have mean zero, so

$$\mathbb{E}[Y_3]=0.$$

Below, first, we calculate the variance of Y_{3a}.

$$\mathrm{Var}(Y_{3a})=\frac{4}{v}\sum_{\substack{i,j,k,\ell(dist)\\ i',j',k',\ell'(dist)}}(\eta_i\widetilde\Omega_{k\ell}\eta_{i'}\widetilde\Omega_{k'\ell'})\cdot\mathbb{E}[W_{jk}^2W_{\ell i}W_{j'k'}^2W_{\ell'i'}].$$

The summand is nonzero only if either the two variables $W_{\ell i}$ and $W_{\ell'i'}$ are the same, or each of the two variables $W_{\ell i}$ and $W_{\ell'i'}$ equals to another squared W term. By (74), (81), and elementary calculations,

$$\begin{aligned}
&(\eta_i\widetilde\Omega_{k\ell}\eta_{i'}\widetilde\Omega_{k'\ell'})\cdot\mathbb{E}[W_{jk}^2W_{\ell i}W_{j'k'}^2W_{\ell'i'}]\\
&\le C\alpha^2\theta_i\theta_k\theta_\ell\theta_{i'}\theta_{k'}\theta_{\ell'}\cdot\mathbb{E}[W_{jk}^2W_{\ell i}W_{j'k'}^2W_{\ell'i'}]\\
&=\begin{cases}
C\alpha^2\theta_i^2\theta_\ell^2\theta_k^2\,\mathbb{E}[W_{jk}^4W_{\ell i}^2]\le C\alpha^2\theta_i^3\theta_j\theta_k^3\theta_\ell^3, & \text{if } \{\ell',i'\}=\{\ell,i\},\ (j',k')=(j,k);\\
C\alpha^2\theta_i^2\theta_\ell^2\theta_k\theta_j\,\mathbb{E}[W_{jk}^4W_{\ell i}^2]\le C\alpha^2\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3, & \text{if } \{\ell',i'\}=\{\ell,i\},\ (j',k')=(k,j);\\
C\alpha^2\theta_i^2\theta_\ell^2\theta_k\theta_{k'}\,\mathbb{E}[W_{jk}^2W_{\ell i}^2W_{j'k'}^2]\le C\alpha^2\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_{j'}\theta_{k'}^2, & \text{if } \{\ell',i'\}=\{\ell,i\},\ \{j',k'\}\neq\{j,k\};\\
C\alpha^2\theta_i^2\theta_\ell\theta_j\theta_k^2\,\mathbb{E}[W_{jk}^3W_{\ell i}^3]\le C\alpha^2\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2, & \text{if } \{\ell',i'\}=\{j,k\},\ (j',k')=(\ell,i);\\
C\alpha^2\theta_i\theta_\ell^2\theta_j\theta_k^2\,\mathbb{E}[W_{jk}^3W_{\ell i}^3]\le C\alpha^2\theta_i^2\theta_j^2\theta_k^3\theta_\ell^3, & \text{if } \{\ell',i'\}=\{j,k\},\ (j',k')=(i,\ell);\\
0, & \text{otherwise.}
\end{cases}
\end{aligned}$$

There are only three different cases in the bounds. It follows that

$$\begin{aligned}
\mathrm{Var}(Y_{3a}) &\le \frac{C\alpha^2}{\|\theta\|_1^2}\Big(\sum_{i,j,k,\ell}\theta_i^3\theta_j\theta_k^3\theta_\ell^3+\sum_{i,j,k,\ell}\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3+\sum_{i,j,k,\ell,j',k'}\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_{j'}\theta_{k'}^2\Big)\\
&\le \frac{C\alpha^2}{\|\theta\|_1^2}\big(\|\theta\|_1\|\theta\|_3^9+\|\theta\|^4\|\theta\|_3^6+\|\theta\|^4\|\theta\|_1^2\|\theta\|_3^6\big)\\
&\le C\alpha^2\|\theta\|^4\|\theta\|_3^6, \qquad (89)
\end{aligned}$$

where in the last line we have used $\|\theta\|_3^9 \le \|\theta\|_3^6(\theta_{\max}\|\theta\|^2) = o(\|\theta\|^2\|\theta\|_3^6)$ and $\|\theta\|_1 \ge \theta_{\max}^{-1}\|\theta\|^2 \to \infty$. Next, we calculate the variance of Y_{3b}. We mimic the argument in (85) and group summands according to the underlying path s-j-k and edge ℓ-i in a complete graph. It yields

$$Y_{3b} = -\frac{2}{\sqrt{v}} \sum_{\substack{\text{length-3}\\ \text{path}}} \sum_{\substack{\text{edge not}\\ \text{in the path}}} \big(\eta_i\widetilde{\Omega}_{k\ell} + \eta_\ell\widetilde{\Omega}_{ki} + \eta_i\widetilde{\Omega}_{s\ell} + \eta_\ell\widetilde{\Omega}_{si}\big) W_{sj}W_{jk}W_{\ell i},$$

where

$$\big|\eta_i\widetilde{\Omega}_{k\ell} + \eta_\ell\widetilde{\Omega}_{ki} + \eta_i\widetilde{\Omega}_{s\ell} + \eta_\ell\widetilde{\Omega}_{si}\big| \le C\alpha(\theta_k + \theta_s)\theta_i\theta_\ell.$$

It follows that

$$\begin{aligned}
\mathrm{Var}(Y_{3b}) &\le \frac{C}{v} \sum_{i,j,k,\ell,s} \alpha^2(\theta_k+\theta_s)^2\theta_i^2\theta_\ell^2 \cdot \mathrm{Var}(W_{sj}W_{jk}W_{\ell i}) \\
&\le \frac{C\alpha^2}{\|\theta\|_1^2} \sum_{i,j,k,\ell,s} (\theta_i^3\theta_j^2\theta_k^3\theta_\ell^3\theta_s + \theta_i^3\theta_j^2\theta_k\theta_\ell^3\theta_s^3) \\
&\le \frac{C\alpha^2\|\theta\|^2\|\theta\|_3^9}{\|\theta\|_1}. \qquad (90)
\end{aligned}$$

Since $\|\theta\|_3^9 \le \|\theta\|_3^6(\theta_{\max}\|\theta\|_1) = o(\|\theta\|_1\|\theta\|_3^6)$, so the right hand side of (90) is much smaller than the right hand side of (89). Together, we have

$$\mathrm{Var}(Y_3) \le C\alpha^2\|\theta\|^4\|\theta\|_3^6 = o(\|\theta\|^8).$$

This proves the claims of Y_3.

Consider Y_4. We plug in $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$ and the expression (77). It gives

$$\begin{aligned}
Y_4 &= \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_j(\eta_i - \tilde{\eta}_i)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} \\
&= \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_j\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq i} W_{is}\Big)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}W_{\ell i} \\
&= -\frac{1}{\sqrt{v}} \sum_{\substack{i,j,\ell(dist)\\ s\neq j}} \Big(\sum_{k\notin\{i,j,\ell\}} \eta_i\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\Big) W_{js}W_{\ell i} - \frac{1}{\sqrt{v}} \sum_{\substack{i,\ell(dist)\\ s\neq i}} \Big(\sum_{j,k\notin\{i,\ell\}} \eta_j\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\Big) W_{is}W_{\ell i} \\
&\equiv Y_{4a} + Y_{4b}.
\end{aligned}$$

First, we analyze Y_{4a}. When (i,j,ℓ) are distinct, $W_{js}W_{\ell i}$ has a mean zero. Therefore,

$$\mathbb{E}[Y_{4a}] = 0.$$

To calculate the variance, we rewrite

$$Y_{4a} = -\frac{1}{\sqrt{v}} \sum_{\substack{i,j,\ell(dist)\\ s\neq j}} \beta_{ij\ell}W_{js}W_{\ell i}, \qquad \text{where} \quad \beta_{ij\ell} = \sum_{k\notin\{i,j,\ell\}} \eta_i\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}$$

By (74) and (81), $|\beta_{ij\ell}| \le C\sum_k \alpha^2\theta_i\theta_j\theta_k^2\theta_\ell \le C\alpha^2\|\theta\|^2\theta_i\theta_j\theta_\ell$. Also, for $W_{js}W_{\ell i}$ and $W_{j's'}W_{\ell' i'}$ to be correlated, there are only two cases: $(W_{js}, W_{\ell i}) = (W_{j's'}, W_{\ell' i'})$ or

$(W_{js}, W_{\ell i}) = (W_{\ell' i'}, W_{j's'})$. Mimicking the argument in (85) or (90), we can easily obtain that

$$
\begin{aligned}
\mathrm{Var}(Y_{4a}) &\leq \frac{C}{v} \sum_{\substack{i,j,\ell (dist)\\ s\neq j}} \beta_{ij\ell}^2 \cdot \mathrm{Var}(W_{js}W_{\ell i}) \\
&\leq \frac{C}{\|\theta\|_1^2} \sum_{i,j,\ell,s} (\alpha^2\|\theta\|^2\theta_i\theta_j\theta_\ell)^2 \cdot \theta_i\theta_j\theta_\ell\theta_s \\
&\leq \frac{C\alpha^4\|\theta\|^4\|\theta\|_3^9}{\|\theta\|_1}. \qquad (91)
\end{aligned}
$$

Next, we analyze Y_{4b}. We re-write

$$
Y_{4b} = -\frac{1}{\sqrt{v}} \sum_{\substack{i,\ell (dist)\\ s\neq i}} \beta_{i\ell} W_{is} W_{\ell i}, \qquad \text{where} \quad \beta_{i\ell} = \sum_{j,k\notin\{i,\ell\}} \eta_j \widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}.
$$

By separating the case of $s=\ell$ from the case of $s\neq\ell$, we have

$$
Y_{4b} = -\frac{1}{\sqrt{v}} \sum_{i,\ell (dist)} \beta_{i\ell} W_{\ell i}^2 - \frac{1}{\sqrt{v}} \sum_{\substack{i,\ell (dist)\\ s\notin\{i,\ell\}}} \beta_{i\ell} W_{is} W_{\ell i} \equiv \widetilde{Y}_{4b} + Y_{4b}^*.
$$

Only $\widetilde{Y}_{4b}$ has a nonzero mean. By (74) and (81),

$$
|\beta_{i\ell}| \leq C\sum_{j,k} \alpha^2\theta_j^2\theta_k^2\theta_\ell \leq C\alpha^2\|\theta\|^4\theta_\ell.
$$

It follows that

$$
|\mathbb{E}[Y_{4b}]| = |\mathbb{E}[\widetilde{Y}_{4b}]| \leq \frac{C}{\|\theta\|_1} \sum_{i,\ell} (\alpha^2\|\theta\|^4\theta_\ell)\theta_i\theta_\ell \leq C\alpha^2\|\theta\|^6. \qquad (92)
$$

We now bound the variances of $\widetilde{Y}_{4b}$ and Y_{4b}^*. By direct calculations,

$$
\mathrm{Var}(\widetilde{Y}_{4b}) = \frac{2}{v} \sum_{i,\ell (dist)} \beta_{i\ell}^2 \cdot \mathrm{Var}(W_{i\ell}^2) \leq \frac{C}{\|\theta\|_1^2} \sum_{i,\ell} (\alpha^2\|\theta\|^4\theta_\ell)^2 \cdot \theta_i\theta_\ell \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1},
$$

$$
\mathrm{Var}(Y_{4b}^*) \leq \frac{C}{v} \sum_{\substack{i,\ell (dist)\\ s\notin\{i,\ell\}}} \beta_{i\ell}^2 \cdot \mathrm{Var}(W_{is}W_{\ell i}) \leq \frac{C}{\|\theta\|_1^2} \sum_{i,\ell,s} (\alpha^2\|\theta\|^4\theta_\ell)^2 \cdot \theta_i^2\theta_\ell\theta_s \leq \frac{C\alpha^4\|\theta\|^{10}\|\theta\|_3^3}{\|\theta\|_1}.
$$

Together, we have

$$
\mathrm{Var}(Y_{4b}) \leq 2\mathrm{Var}(\widetilde{Y}_{4b}) + 2\mathrm{Var}(Y_{4b}^*) \leq \frac{C\alpha^4\|\theta\|^{10}\|\theta\|_3^3}{\|\theta\|_1}. \qquad (93)
$$

We combine the results of Y_{4a} and Y_{4b}. Since $\|\theta\|_3^6 \leq (\theta_{\max}\|\theta\|^2)^2 = o(\|\theta\|^4)$, the right hand side of (92) dominates the right hand side of (91). It follows that

$$
|\mathbb{E}[Y_4]| \leq C\alpha^2\|\theta\|^6 = o(\alpha^4\|\theta\|^8), \quad \mathrm{Var}(Y_4) \leq \frac{C\alpha^4\|\theta\|^{10}\|\theta\|_3^3}{\|\theta\|_1} = o(\alpha^6\|\theta\|^8\|\theta\|_3^6).
$$

Here, we explain the equalities. The first one is due to $\alpha^2\|\theta\|^2 \to \infty$. To get the second equality, we compare $\mathrm{Var}(Y_4)$ with the order of $\alpha^6\|\theta\|^8\|\theta\|_3^6$. Note that $\frac{\|\theta\|^{10}\|\theta\|_3^3}{\|\theta\|_1} = \frac{\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}\|\theta\|^4 \leq$

$\frac{\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}\|\theta\|_1\|\theta\|_3^3 \le \|\theta\|^6\|\theta\|_3^6$. It follows that $\mathrm{Var}(Y_4) \le C\alpha^4\|\theta\|^6\|\theta\|_3^6 \ll C\alpha^6\|\theta\|^8\|\theta\|_3^6$, where the last inequality is due to $\alpha^2\|\theta\|^2 \to \infty$. So far, we have proved all claims about Y_4.

Consider Y_5. Recall that

$$Y_5 = \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}W_{k\ell}\widetilde{\Omega}_{\ell i} + \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j\widetilde{\Omega}_{jk}W_{k\ell}\widetilde{\Omega}_{\ell i}.$$

With relabeling of $(i,j,k,\ell) = (j',i',\ell',k')$, the second sum can be written as $\sum_{i',j',k',\ell'(dist)}(\eta_{j'} - \tilde{\eta}_{j'})\eta_{i'}\widetilde{\Omega}_{i'\ell'}W_{\ell'k'}\widetilde{\Omega}_{k'j'}$. This suggests that it is actually equal to the first sum above. Hence,

$$\begin{aligned}
Y_5 &= 2\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}W_{k\ell}\widetilde{\Omega}_{\ell i} \\
&= \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{2}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)\widetilde{\Omega}_{jk}W_{k\ell}\widetilde{\Omega}_{\ell i} \\
&= -\frac{2}{\sqrt{v}}\sum_{\substack{j,k,\ell(dist)\\ s\neq j}}\Big(\sum_{i\notin\{j,k,\ell\}} \eta_i\widetilde{\Omega}_{jk}\widetilde{\Omega}_{\ell i}\Big)W_{js}W_{k\ell} \\
&\equiv -\frac{2}{\sqrt{v}}\sum_{\substack{j,k,\ell(dist)\\ s\neq j}} \beta_{jk\ell}W_{js}W_{k\ell}, \qquad \text{where} \quad \beta_{jk\ell} \equiv \sum_{i\notin\{j,k,\ell\}} \eta_i\widetilde{\Omega}_{jk}\widetilde{\Omega}_{\ell i}.
\end{aligned}$$

It is easy to see that $\mathbb{E}[W_{js}W_{k\ell}] = 0$ when (j,k,ℓ) are distinct. Hence,

$$\mathbb{E}[Y_5] = 0.$$

By (74) and (81), $|\beta_{jk\ell}| \le C\sum_i \theta_i \cdot \alpha^2\theta_j\theta_k\theta_\ell\theta_i \le C\alpha^2\|\theta\|^2\theta_j\theta_k\theta_\ell$. Similar to the argument in (85) or (90), we can show that

$$\begin{aligned}
\mathrm{Var}(Y_5) &\le \frac{C}{v}\sum_{\substack{j,k,\ell(dist)\\ s\neq j}} \beta_{jk\ell}^2 \cdot \mathrm{Var}(W_{js}W_{k\ell}) \\
&\le \frac{C}{\|\theta\|_1^2}\sum_{j,k,\ell,s} (\alpha^2\|\theta\|^2\theta_j\theta_k\theta_\ell)^2\theta_j\theta_s\theta_k\theta_\ell \\
&\le \frac{C\alpha^4\|\theta\|^4\|\theta\|_3^9}{\|\theta\|_1}.
\end{aligned}$$

Since $\|\theta\|_3^9 = (\|\theta\|_3^3)^2\|\theta\|_3^3 \le (\theta_{\max}\|\theta\|^2)^2(\theta_{\max}^2\|\theta\|_1) = o(\|\theta\|^4\|\theta\|_1)$, the right hand side is $o(\|\theta\|^8)$. This proves the claims of Y_5.

Consider Y_6. By definition and elementary calculations,

$$\begin{aligned}
Y_6 &= \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_j(\eta_i - \tilde{\eta}_i)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} \\
&= 2\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} \\
&= 2\sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} \\
&= -\frac{2}{\sqrt{v}}\sum_{j,s(dist)}\Big(\sum_{i,k,\ell(dist)\notin\{j\}} \eta_i\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}\Big)W_{js}.
\end{aligned}$$

Here, to get the second line above, we relabeled $(i,j,k,\ell) = (j',i',\ell',k')$ in the second sum and found out the two sums are equal; the third line is from (77). We immediately see that

$$\mathrm{E}[Y_6] = 0.$$

By (74) and (81),

$$\Big|\sum_{i,k,\ell(dist)\notin\{j\}} \eta_i \widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}\Big| \leq \sum_{i,k,\ell} C\theta_i \cdot \alpha^3\theta_j\theta_k^2\theta_\ell^2\theta_i \leq C\alpha^3\|\theta\|^6\theta_j.$$

It follows that

$$\begin{aligned}\mathrm{Var}(Y_6) &= \frac{8}{v}\sum_{j,s(dist)}\Big(\sum_{i,k,\ell(dist)\notin\{j\}} \eta_i \widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}\Big)^2 \cdot \mathrm{Var}(W_{js})\\ &\leq \frac{C}{\|\theta\|_1^2}\sum_{j,s}(\alpha^3\|\theta\|^6\theta_j)^2\theta_j\theta_s\\ &\leq \frac{C\alpha^6\|\theta\|^{12}\|\theta\|_3^3}{\|\theta\|_1}.\end{aligned}$$

Since $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, the variance is bounded by $C\alpha^6\|\theta\|^8\|\theta\|_3^6$. This proves the claims of Y_6.

G.4.6. *Proof of Lemma G.6.* It suffices to prove the claims for each of Z_1 and Z_2; then, the claims of U_b follow immediately.

We first analyze Z_1. Plugging $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$ into the definition of Z_1 gives

$$\begin{aligned}Z_1 = &\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\eta_j(\eta_k - \tilde{\eta}_k)W_{k\ell}W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)^2\eta_k W_{k\ell}W_{\ell i}\\ &+ \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j^2(\eta_k - \tilde{\eta}_k)W_{k\ell}W_{\ell i} + \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j(\eta_j - \tilde{\eta}_j)\eta_k W_{k\ell}W_{\ell i}.\end{aligned}$$

In the last term above, if we relabel $(i,j,k,\ell) = (k',j',i',\ell')$, it becomes $\sum_{i',j',k',\ell'(dist)}(\eta_{k'} - \tilde{\eta}_{k'})\eta_{j'}(\eta_{j'} - \tilde{\eta}_{j'})\eta_{i'}W_{i'\ell'}W_{\ell'k'}$. This shows that the last sum equals to the first sum. Therefore,

$$\begin{aligned}Z_1 = &\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)^2\eta_k W_{k\ell}W_{\ell i}\\ &+2\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\eta_j(\eta_k - \tilde{\eta}_k)W_{k\ell}W_{\ell i}\\ &+\sum_{i,j,k,\ell(dist)} (\tilde{\eta}_i - \eta_i)\eta_j^2(\tilde{\eta}_k - \eta_k)W_{k\ell}W_{\ell i}\\ \equiv\ & Z_{1a} + Z_{1b} + Z_{1c}.\end{aligned} \tag{94}$$

Below, we compute the means and variances of Z_{1a}-Z_{1c}.

First, we study Z_{1a}. When (i,j,k,ℓ) are distinct, $W_{k\ell}W_{\ell i}$ has a mean zero and is independent of $(\tilde{\eta}_j - \eta_j)^2$, so $\mathbb{E}[(\eta_j - \tilde{\eta}_j)^2 W_{k\ell}W_{\ell i}] = 0$. It follows that

$$\mathbb{E}[Z_{1a}] = 0.$$

To bound the variance of Z_{1a}, we use (77) to re-write

$$Z_{1a} = \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq j} W_{jt}\Big)\eta_k W_{k\ell}W_{\ell i}$$

$$
\begin{aligned}
&= \frac{1}{v} \sum_{\substack{i,j,k,\ell(dist)\\ s,t\notin\{j\}}} \eta_i\eta_k W_{js}W_{jt}W_{k\ell}W_{\ell i} \\
&= \frac{1}{v} \sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{j\}}} \eta_i\eta_k W_{js}^2 W_{k\ell}W_{\ell i} + \frac{1}{v} \sum_{\substack{i,j,k,\ell(dist)\\ s,t(dist)\notin\{j\}}} \eta_i\eta_k W_{js}W_{jt}W_{k\ell}W_{\ell i} \\
&\equiv \widetilde{Z}_{1a} + Z_{1a}^*.
\end{aligned}
$$

We first bound the variance of $\widetilde{Z}_{1a}$. It is seen that

$$
\mathrm{Var}(\widetilde{Z}_{1a}) = \frac{1}{v^2} \sum_{\substack{i,j,k,\ell(dist),s\notin\{j\}\\ i',j',k',\ell'(dist),s'\notin\{j'\}}} \eta_i\eta_k\eta_{i'}\eta_{k'} \cdot \mathbb{E}[W_{js}^2 W_{k\ell}W_{\ell i} \cdot W_{j's'}^2 W_{k'\ell'}W_{\ell' i'}].
$$

The summand is nonzero only if $\ell' = \ell$ and $\{k', i'\} = \{k, i\}$. We also note that, if we switch i' and k', the summand remains unchanged. So, it suffices to consider the case of $\ell' = \ell$ and $(k', i') = (k, i)$. By (81) and elementary calculations,

$$
\begin{aligned}
&\eta_i\eta_k\eta_{i'}\eta_{k'} \cdot \mathbb{E}[W_{js}^2 W_{k\ell}W_{\ell i} \cdot W_{j's'}^2 W_{k'\ell'}W_{\ell' i'}] \\
&= \begin{cases}
\eta_i^2\eta_k^2\,\mathbb{E}[W_{js}^4 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_s, & \text{if } (\ell', k', i') = (\ell, k, i),\ \{j', s'\} = \{j, s\}; \\
\eta_i^2\eta_k^2\,\mathbb{E}[W_{js}^2 W_{k\ell}^2 W_{\ell i}^2 W_{j's'}^2] \le C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_s\theta_{j'}\theta_{s'}, & \text{if } (\ell', k', i') = (\ell, k, i),\ \{j', s'\} \neq \{j, s\}; \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathrm{Var}(\widetilde{Z}_{1a}) &\le \frac{C}{\|\theta\|_1^4} \Big(\sum_{i,j,k,\ell,s} \theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_s + \sum_{i,j,k,\ell,s,j',s'} \theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_s\theta_{j'}\theta_{s'} \Big) \\
&\le \frac{C}{\|\theta\|_1^4} \big(\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^2\|\theta\|_3^6\|\theta\|_1^4 \big) \\
&\le C\|\theta\|^2\|\theta\|_3^6.
\end{aligned}
$$

We then bound the variance of Z_{1a}^*. Note that

$$
\begin{aligned}
&\eta_i\eta_k\eta_{i'}\eta_{k'} \cdot \mathbb{E}[W_{js}W_{jt}W_{k\ell}W_{\ell i} \cdot W_{j's'}W_{j't'}W_{k'\ell'}W_{\ell' i'}] \\
&= \begin{cases}
\eta_i^2\eta_k^2\,\mathbb{E}[W_{js}^2 W_{jt}^2 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2\theta_s\theta_t, & \text{if } (j', \ell') = (j, \ell), \{s', t'\} = \{s, t\}, \{k', i'\} = \{k, i\}; \\
\eta_i\eta_k\eta_s\eta_t\,\mathbb{E}[W_{js}^2 W_{jt}^2 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s^2\theta_t^2, & \text{if } (j', \ell') = (\ell, j), \{s', t'\} = \{k, i\}, \{k', i'\} = \{s, t\}; \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathrm{Var}(Z_{1a}^*) &\le \frac{C}{\|\theta\|_1^4} \Big(\sum_{i,j,k,\ell,s,t} \theta_i^3\theta_j^2\theta_k^3\theta_\ell^2\theta_s\theta_t + \sum_{i,j,k,\ell,s,t} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s^2\theta_t^2 \Big) \\
&\le \frac{C}{\|\theta\|_1^4} \big(\|\theta\|^4\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^{12} \big) \\
&\le \frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2},
\end{aligned}
$$

where the last inequality is because of $\|\theta\|^{12} = \|\theta\|^4(\|\theta\|^4)^2 \le \|\theta\|^4(\|\theta\|_1\|\theta\|_3^3)^2 = \|\theta\|^4\|\theta\|_3^6\|\theta\|_1^2$. Combining the above gives

$$
\mathrm{Var}(Z_{1a}) \le 2\mathrm{Var}(\widetilde{Z}_{1a}) + 2\mathrm{Var}(Z_{1a}^*) \le C\|\theta\|^2\|\theta\|_3^6. \tag{95}
$$

Second, we study Z_{1b}. Since $(\eta_j - \tilde{\eta}_j)$, $(\eta_k - \tilde{\eta}_k)W_{k\ell}$ and $W_{\ell i}$ are independent of each other, each summand in Z_{1b} has a zero mean. It follows that

$$\mathbb{E}[Z_{1b}] = 0.$$

We now compute its variance. By direct calculations,

$$\begin{aligned}
Z_{1b} &= 2 \sum_{i,j,k,\ell (dist)} \eta_i \Big(-\frac{1}{\sqrt{v}} \sum_{s\neq j} W_{js}\Big) \eta_j \Big(-\frac{1}{\sqrt{v}} \sum_{t\neq k} W_{kt}\Big) W_{k\ell} W_{\ell i} \\
&= \frac{2}{v} \sum_{\substack{i,j,k,\ell (dist) \\ s\neq j, t\neq k}} \eta_i \eta_j W_{js} W_{kt} W_{k\ell} W_{\ell i} \\
&= \frac{2}{v} \sum_{\substack{i,j,k,\ell (dist) \\ s\neq j}} \eta_i \eta_j W_{js} W_{k\ell}^2 W_{\ell i} + \frac{2}{v} \sum_{\substack{i,j,k,\ell (dist) \\ s\neq j, t\notin \{k,\ell\}}} \eta_i \eta_j W_{js} W_{kt} W_{k\ell} W_{\ell i} \\
&\equiv \widetilde{Z}_{1b} + Z_{1b}^*.
\end{aligned}$$

We first bound the variance of $\widetilde{Z}_{1b}$. Note that

$$\mathrm{Var}(\widetilde{Z}_{1b}) = \frac{4}{v} \sum_{\substack{i,j,k,\ell (dist), s\neq j \\ i',j',k',\ell' (dist), s'\neq j'}} \eta_i \eta_j \eta_{i'} \eta_{j'} \cdot \mathbb{E}[W_{js} W_{k\ell}^2 W_{\ell i} \cdot W_{j's'} W_{k'\ell'}^2 W_{\ell' i'}].$$

For this summand to be nonzero, there are only two cases. In the first case, $(W_{js}, W_{\ell i})$ are paired with $(W_{j's'}, W_{\ell' i'})$. It follows that

$$\eta_i \eta_j \eta_{i'} \eta_{j'} \cdot \mathbb{E}[W_{js} W_{k\ell}^2 W_{\ell i} W_{j's'} W_{k'\ell'}^2 W_{\ell' i'}] = \eta_i \eta_j \eta_{i'} \eta_{j'} \cdot \mathbb{E}[W_{js}^2 W_{k\ell}^2 W_{\ell i}^2 W_{k'\ell'}^2].$$

This happens only if (i) $\{j', s'\} = \{j, s\}$ and $\{\ell', i'\} = \{\ell, i\}$, or (ii) $\{j', s'\} = \{\ell, i\}$ and $\{\ell', i'\} = \{j, s\}$. By (81) and elementary calculations,

$$\begin{aligned}
&\eta_i \eta_j \eta_{i'} \eta_{j'} \cdot \mathbb{E}[W_{js} W_{k\ell}^2 W_{\ell i} \cdot W_{j's'} W_{k'\ell'}^2 W_{\ell' i'}] \\
&= \begin{cases}
\eta_i^2 \eta_j^2 \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'\ell}^2] \leq C\theta_i^3 \theta_j^3 \theta_k \theta_\ell^3 \theta_s \theta_{k'}, & \text{if } (j', s') = (j, s), (\ell', i') = (\ell, i); \\
\eta_i \eta_j^2 \eta_\ell \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'i}^2] \leq C\theta_i^3 \theta_j^3 \theta_k \theta_\ell^3 \theta_s \theta_{k'}, & \text{if } (j', s') = (j, s), (\ell', i') = (i, \ell); \\
\eta_i^2 \eta_j \eta_s \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'\ell}^2] \leq C\theta_i^3 \theta_j^2 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (s, j), (\ell', i') = (\ell, i); \\
\eta_i \eta_j \eta_\ell \eta_s \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'i}^2] \leq C\theta_i^3 \theta_j^2 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (s, j), (\ell', i') = (i, \ell); \\
\eta_i \eta_j \eta_\ell \eta_s \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'j}^2] \leq C\theta_i^2 \theta_j^3 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (\ell, i), (\ell', i') = (j, s); \\
\eta_i \eta_j^2 \eta_\ell \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k's}^2] \leq C\theta_i^2 \theta_j^3 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (\ell, i), (\ell', i') = (s, j); \\
\eta_i^2 \eta_j \eta_s \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k'j}^2] \leq C\theta_i^3 \theta_j^3 \theta_k \theta_\ell^2 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (i, \ell), (\ell', i') = (j, s); \\
\eta_i^2 \eta_j^2 \cdot \mathbb{E}[W_{js}^2 W_{\ell i}^2 W_{k\ell}^2 W_{k's}^2] \leq C\theta_i^3 \theta_j^3 \theta_k \theta_\ell^2 \theta_s^2 \theta_{k'}, & \text{if } (j', s') = (i, \ell), (\ell', i') = (s, j); \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}$$

The upper bound on the right hand side only has two types $C\theta_i^3 \theta_j^3 \theta_k \theta_\ell^3 \theta_s \theta_{k'}$ and $C\theta_i^3 \theta_j^2 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'}$. The contribution of this case to $\mathrm{Var}(\widetilde{Z}_{1b})$ is

$$\begin{aligned}
&\leq \frac{C}{v^2} \Big(\sum_{i,j,k,\ell,s,k'} \theta_i^3 \theta_j^3 \theta_k \theta_\ell^3 \theta_s \theta_{k'} + \sum_{i,j,k,\ell,s,k'} \theta_i^3 \theta_j^2 \theta_k \theta_\ell^3 \theta_s^2 \theta_{k'} \Big) \\
&\leq \frac{C}{\|\theta\|_1^4} \big(\|\theta\|_3^9 \|\theta\|_1^3 + \|\theta\|^4 \|\theta\|_3^6 \|\theta\|_1^2 \big)
\end{aligned}$$

$$\le \frac{C\|\theta\|_3^9}{\|\theta\|_1}.$$

In the second case, $\{W_{js}, W_{k\ell}, W_{\ell i}\}$ and $\{W_{j's'}, W_{k'\ell'}, W_{\ell' i'}\}$ are two sets of same variables. Then,

$$\eta_i\eta_j\eta_{i'}\eta_{j'} \cdot \mathbb{E}[W_{js}W_{k\ell}^2 W_{\ell i} W_{j's'} W_{k'\ell'}^2 W_{\ell' i'}] = \eta_i\eta_j\eta_{i'}\eta_{j'} \cdot \mathbb{E}[W_{js}^3 W_{k\ell}^3 W_{\ell i}^3].$$

This can only happen if $\ell' = \ell$, $\{i', k'\} = \{i, k\}$, and $\{j', s'\} = \{j, s\}$. By (81) and elementary calculations,

$$\eta_i\eta_j\eta_{i'}\eta_{j'} \cdot \mathbb{E}[W_{js}W_{k\ell}^2 W_{\ell i} \cdot W_{j's'} W_{k'\ell'}^2 W_{\ell' i'}]$$
$$= \begin{cases} \eta_i^2\eta_j^2 \cdot \mathbb{E}[W_{js}^3 W_{\ell i}^3 W_{k\ell}^3] \le C\theta_i^3\theta_j^3\theta_k\theta_\ell^2\theta_s, & \text{if } \ell' = \ell,\ (i', k') = (i, k),\ (j', s') = (j, s); \\ \eta_i^2\eta_j\eta_s \cdot \mathbb{E}[W_{js}^3 W_{\ell i}^3 W_{k\ell}^3] \le C\theta_i^3\theta_j^2\theta_k\theta_\ell^2\theta_s^2, & \text{if } \ell' = \ell,\ (i', k') = (i, k),\ (j', s') = (s, j); \\ \eta_i\eta_k\eta_j^2 \cdot \mathbb{E}[W_{js}^3 W_{\ell i}^3 W_{k\ell}^3] \le C\theta_i^2\theta_j^3\theta_k^2\theta_\ell^2\theta_s, & \text{if } \ell' = \ell,\ (i', k') = (k, i),\ (j', s') = (j, s); \\ \eta_i\eta_k\eta_j\eta_s \cdot \mathbb{E}[W_{js}^3 W_{\ell i}^3 W_{k\ell}^3] \le C\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s^2, & \text{if } \ell' = \ell,\ (i', k') = (i, k),\ (j', s') = (s, j); \\ 0, & \text{otherwise.} \end{cases}$$

The upper bound on the right hand side has three types, and the contribution of this case to $\mathrm{Var}(\widetilde{Z}_{1b})$ is

$$\le \frac{C}{v^2}\Big(\sum_{i,j,k,\ell,s} \theta_i^3\theta_j^3\theta_k\theta_\ell^2\theta_s + \sum_{i,j,k,\ell,s} \theta_i^3\theta_j^2\theta_k\theta_\ell^2\theta_s^2 + \sum_{i,j,k,\ell,s} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s^2\Big)$$
$$\le \frac{C}{\|\theta\|_1^4}\big(\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^6\|\theta\|_3^3\|\theta\|_1 + \|\theta\|^{10}\big)$$
$$\le \frac{C\|\theta\|^2\|\theta\|_3^6}{\|\theta\|_1^2},$$

where we use $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$ (Cauchy-Schwarz) in the last line. It is seen that the contribution of the first case is dominating, and so

$$\mathrm{Var}(\widetilde{Z}_{1b}) \le \frac{C\|\theta\|_3^9}{\|\theta\|_1}.$$

We then bound the variance of Z_{1b}^*. Note that

$$\mathrm{Var}(Z_{1b}^*) = \frac{4}{v^2} \sum_{\substack{i,j,k,\ell (dist), s\neq j, t\notin\{k,\ell\} \\ i',j',k',\ell' (dist), s'\neq j', t'\notin\{k',\ell'\}}} \eta_i\eta_j\eta_{i'}\eta_{j'} \cdot \mathbb{E}[W_{js}W_{kt}W_{k\ell}W_{\ell i} \cdot W_{j's'}W_{k't'}W_{k'\ell'}W_{\ell' i'}].$$

For the summand to be nonzero, all W terms have to be perfectly matched, so that the expectation in the summand becomes

$$\mathbb{E}[W_{js}W_{kt}W_{k\ell}W_{\ell i} \cdot W_{j's'}W_{k't'}W_{k'\ell'}W_{\ell' i'}] = \mathbb{E}[W_{js}^2 W_{kt}^2 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i\theta_j\theta_k^2\theta_\ell^2\theta_s\theta_t.$$

For this perfect match to happen, we need $(t', k', \ell', i') = (t, k, \ell, i)$ or $(t', k', \ell', i') = (i, \ell, k, t)$, as well as $\{j', s'\} = \{j, s\}$. This implies that, i' can only take values in $\{i, t\}$ and j' can only take values in $\{j, s\}$. It follows that $\eta_i\eta_j\eta_{i'}\eta_{j'}$ belongs to one of the following cases:

$$\eta_i\eta_j(\eta_i\eta_j) \le C\theta_i^2\theta_j^2, \qquad \eta_i\eta_j(\eta_i\eta_s) = C\theta_i^2\theta_j\theta_s,$$
$$\eta_i\eta_j(\eta_t\eta_j) \le C\theta_i\theta_j^2\theta_t, \qquad \eta_i\eta_j(\eta_t\eta_s) \le C\theta_i\theta_j\theta_t\theta_s.$$

Combining the above gives

$$\begin{aligned}\mathrm{Var}(Z_{1b}^*) &\le \frac{C}{v^2}\sum_{i,j,k,\ell,s,t}(\theta_i^2\theta_j^2+\theta_i^2\theta_j\theta_s+\theta_i\theta_j^2\theta_t+\theta_i\theta_j\theta_t\theta_s)\cdot\theta_i\theta_j\theta_k^2\theta_\ell^2\theta_s\theta_t\\ &\le \frac{C}{\|\theta\|_1^4}\big(\|\theta\|^4\|\theta\|_3^6\|\theta\|_1^2+2\|\theta\|^8\|\theta\|_3^3\|\theta\|_1+\|\theta\|^{12}\big)\\ &\le \frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2}.\end{aligned}$$

We combine the variances of $\widetilde{Z}_{1b}$ and Z_{1b}^*. Since $\|\theta\|^4\le\|\theta\|_1\|\theta\|_3^3$, the variance of $\widetilde{Z}_{1b}$ dominates. It follows that

$$\mathrm{Var}(Z_{1b})\le 2\mathrm{Var}(\widetilde{Z}_{1b})+2\mathrm{Var}(Z_{1b}^*)\le\frac{C\|\theta\|_3^9}{\|\theta\|_1}. \tag{96}$$

Third, we study Z_{1c}. It is seen that

$$\begin{aligned}Z_{1c} &= \sum_{i,j,k,\ell(dist)}\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq i}W_{is}\Big)\eta_j^2\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq k}W_{kt}\Big)W_{k\ell}W_{\ell i}\\ &= \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\neq i,t\neq k}}\Big(\sum_{j\notin\{i,k,\ell\}}\eta_j^2\Big)W_{is}W_{kt}W_{k\ell}W_{\ell i}\\ &\equiv \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\neq i,t\neq k}}\beta_{ik\ell}W_{is}W_{kt}W_{k\ell}W_{\ell i},\end{aligned}$$

where

$$\beta_{ik\ell}\equiv\sum_{j\notin\{i,k,\ell\}}\eta_j^2\le C\sum_j\theta_j^2\le C\|\theta\|^2. \tag{97}$$

We divide all summands into four groups: (i) $s=t=\ell$; (ii) $s=\ell$, $t\neq\ell$; (iii) $s\neq\ell$, $t=\ell$; (iv) $s\neq\ell$, $t\neq\ell$. It yields that

$$\begin{aligned}Z_{1c} &= \frac{1}{v}\sum_{i,k,\ell(dist)}\beta_{ik\ell}W_{k\ell}^2W_{\ell i}^2+\frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ t\neq\{k,\ell\}}}\beta_{ik\ell}W_{kt}W_{k\ell}W_{\ell i}^2\\ &\quad+\frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\}}}\beta_{ik\ell}W_{is}W_{k\ell}^2W_{\ell i}+\frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\},t\notin\{k,\ell\}}}\beta_{ik\ell}W_{is}W_{kt}W_{k\ell}W_{\ell i}.\end{aligned}$$

In the third sum, if we relabel $(i,k,\ell,s)=(k',i',\ell',t')$, it has the form $\sum_{i',k',\ell'(dist),t'\notin\{k',\ell'\}}\beta_{k'i'\ell'}W_{k't'}W_{i'\ell'}^2W_{\ell'k'}$. This shows that this sum equals to the second sum. We thus have

$$\begin{aligned}Z_{1c} &= \frac{1}{v}\sum_{i,k,\ell(dist)}\beta_{ik\ell}W_{k\ell}^2W_{\ell i}^2+\frac{2}{v}\sum_{\substack{i,k,\ell(dist)\\ t\neq\{k,\ell\}}}\beta_{ik\ell}W_{kt}W_{k\ell}W_{\ell i}^2\\ &\quad+\frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\},t\notin\{k,\ell\}}}\beta_{ik\ell}W_{is}W_{kt}W_{k\ell}W_{\ell i}\\ &\equiv \widetilde{Z}_{1c}+Z_{1c}^*+Z_{1c}^\dagger.\end{aligned}$$

Among all three terms, only $\widetilde{Z}_{1c}$ has a nonzero mean. It follows that

$$\mathbb{E}[Z_{1c}] = \mathbb{E}[\widetilde{Z}_{1c}] = \frac{1}{v} \sum_{i,k,\ell(dist)} \beta_{ik\ell}\Omega_{k\ell}(1-\Omega_{k\ell})\Omega_{\ell i}(1-\Omega_{\ell i})$$
$$= \frac{1}{v} \sum_{i,k,\ell(dist)} \beta_{ik\ell}\Omega_{k\ell}\Omega_{\ell i}[1+O(\theta_{\max}^2)].$$

Under the null hypothesis, $\Omega_{ij} = \theta_i\theta_j$. It follows that $\eta_j = \frac{\theta_j}{\sqrt{v}}\sum_{i:i\neq j}\theta_i = [1+o(1)]\frac{\theta_j\|\theta\|_1}{\sqrt{v}}$ and that $\beta_{ik\ell} = \sum_{j\notin\{i,k,\ell\}}\eta_j^2 = [1+o(1)]\frac{\|\theta\|_1^2}{v}\sum_{j\notin\{i,k,\ell\}}\theta_j^2 = [1+o(1)]\frac{\|\theta\|_1^2\|\theta\|^2}{v}$. Additionally, $v = \sum_{i\neq j}\theta_i\theta_j = \|\theta\|_1^2\cdot[1+o(1)]$. As a result,

$$\mathbb{E}[Z_{1c}] = \frac{1}{v}\sum_{i,k,\ell(dist)}[1+o(1)]\frac{\|\theta\|_1^2\|\theta\|^2}{v}\cdot\theta_k\theta_\ell^2\theta_i$$
$$= [1+o(1)]\cdot\frac{\|\theta\|_1^2\|\theta\|^2}{v^2}\sum_{i,k,\ell(dist)}\theta_k\theta_\ell^2\theta_i$$
$$= [1+o(1)]\cdot\frac{\|\theta\|_1^2\|\theta\|^2}{\|\theta\|_1^4}\big[\|\theta\|_1^2\|\theta\|^2 - O(\|\theta\|^4 + \|\theta\|_1\|\theta\|_3^3)\big]$$
$$= [1+o(1)]\cdot\|\theta\|^4, \tag{98}$$

where in the last line we have used $\|\theta\|^2 = o(\|\theta\|_1)$, $\|\theta\|_3^3 = o(\|\theta\|_1)$ and $\|\theta\|_1\to\infty$. We then bound the variance of Z_{1c} by studying the variance of each of the three variables, $\widetilde{Z}_{1c}$, Z_{1c}^* and $Z_{1c}^\dagger$. Consider $\widetilde{Z}_{1c}$ first. For $W_{k\ell}^2W_{\ell i}^2$ and $W_{k'\ell'}^2W_{\ell'i'}^2$ to be correlated, it has to be the case of either $\{k',\ell'\}=\{k,\ell\}$ or $\{i',\ell'\}=\{i,\ell\}$. By symmetry between k and i in the expression, it suffices to consider $\{k',\ell'\}=\{k,\ell\}$. Direct calculations show that

$$\mathrm{Cov}(W_{k\ell}^2W_{\ell i}^2,\, W_{k'\ell'}^2W_{\ell'i'}^2) \le \begin{cases} \mathbb{E}[W_{k\ell}^4W_{\ell i}^4] \le C\theta_k\theta_\ell^2\theta_i, & \text{if } (k',\ell')=(k,\ell),\ i'=i;\\ \mathbb{E}[W_{k\ell}^4W_{\ell i}^2W_{\ell i'}^2] \le C\theta_k\theta_\ell^3\theta_i\theta_{i'}, & \text{if } (k',\ell')=(k,\ell),\ i'\neq i;\\ \mathbb{E}[W_{k\ell}^4W_{\ell i}^2W_{ki}^2] \le C\theta_k^2\theta_\ell^2\theta_i^2, & \text{if } (k',\ell')=(\ell,k),\ i'=i;\\ \mathbb{E}[W_{k\ell}^4W_{\ell i}^2W_{ki'}^2] \le C\theta_k^2\theta_\ell^2\theta_i\theta_{i'}, & \text{if } (k',\ell')=(\ell,k),\ i'\neq i;\\ 0, & \text{otherwise.}\end{cases}$$

Combining it with (97) and the fact of $v\ge C^{-1}\|\theta\|_1^2$, we have

$$\mathrm{Var}(\widetilde{Z}_{1c}) \le \frac{C\|\theta\|^4}{\|\theta\|_1^4}\Big(\sum_{i,k,\ell}\theta_k\theta_\ell^2\theta_i + \sum_{i,k,\ell,i'}\theta_k\theta_\ell^3\theta_i\theta_{i'} + \sum_{i,k,\ell}\theta_k^2\theta_\ell^2\theta_i^2 + \sum_{i,k,\ell,i'}\theta_k^2\theta_\ell^2\theta_i\theta_{i'}\Big)$$
$$\le \frac{C\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|^2\|\theta\|_1^2 + \|\theta\|_3^3\|\theta\|_1^3 + \|\theta\|^6 + \|\theta\|^4\|\theta\|_1^2\big)$$
$$\le \frac{C\|\theta\|^4\|\theta\|_3^3}{\|\theta\|_1}.$$

Consider Z_{1c}^*. By direct calculations,

$$\mathbb{E}[W_{kt}W_{k\ell}W_{\ell i}^2W_{k't'}W_{k'\ell'}W_{\ell'i'}^2]$$

$$= \begin{cases} \mathbb{E}[W_{kt}^2 W_{k\ell}^2 W_{\ell i}^4] \le C\theta_i\theta_k^2\theta_\ell^2\theta_t, & \text{if } (k',t',\ell') = (k,t,\ell),\ i = i'; \\ \mathbb{E}[W_{kt}^2 W_{k\ell}^2 W_{\ell i}^2 W_{\ell i'}^2] \le C\theta_i\theta_k^2\theta_\ell^3\theta_t\theta_{i'}, & \text{if } (k',t',\ell') = (k,t,\ell),\ i \neq i'; \\ \mathbb{E}[W_{kt}^2 W_{k\ell}^2 W_{\ell i}^2 W_{ti'}^2] \le C\theta_i\theta_k^2\theta_\ell^2\theta_t^2\theta_{i'}, & \text{if } (k',t',\ell') = (k,\ell,t); \\ \mathbb{E}[W_{kt}^3 W_{k\ell}^2 W_{\ell i}^3] \le C\theta_i\theta_k^2\theta_\ell^2\theta_t, & \text{if } (k',t',\ell',i') = (\ell,i,k,t); \\ 0, & \text{otherwise.} \end{cases}$$

We combine it with (97) and find that

$$\begin{aligned} \mathrm{Var}(Z_{1c}^*) &= \frac{4}{v^2} \sum_{\substack{i,k,\ell(dist),t\neq\{k,\ell\} \\ i',k',\ell'(dist),t'\neq\{k',\ell'\}}} \beta_{ik\ell}\beta_{i'k'\ell'} \cdot \mathbb{E}[W_{kt}W_{k\ell}W_{\ell i}^2 W_{k't'}W_{k'\ell'}W_{\ell' i'}^2] \\ &\le \frac{C\|\theta\|^4}{\|\theta\|_1^4}\Big(\sum_{i,k,\ell,t} \theta_i\theta_k^2\theta_\ell^2\theta_t + \sum_{i,k,\ell,t,i'} \theta_i\theta_k^2\theta_\ell^3\theta_t\theta_{i'} + \sum_{i,k,\ell,t,i'} \theta_i\theta_k^2\theta_\ell^2\theta_t^2\theta_{i'} \Big) \\ &\le \frac{C\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|^4\|\theta\|_1^2 + \|\theta\|^2\|\theta\|_3^3\|\theta\|_1^3 + \|\theta\|^6\|\theta\|_1^2 \big) \\ &\le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}. \end{aligned}$$

Consider $Z_{1c}^\dagger$. Re-write

$$Z_{1c}^\dagger = \frac{1}{v}\sum_{i,k,\ell(dist)} \beta_{ik\ell}W_{ik}^2 W_{k\ell}W_{\ell i} + \frac{1}{v}\sum_{\substack{i,k,\ell(dist) \\ s\notin\{i,\ell\},t\notin\{k,\ell\} \\ (s,t)\neq(k,i)}} \beta_{ik\ell}W_{is}W_{kt}W_{k\ell}W_{\ell i}.$$

Regarding the first term, by direct calculations,

$$\begin{aligned} &\mathbb{E}[W_{ik}^2 W_{k\ell}W_{\ell i} \cdot W_{i'k'}^2 W_{k'\ell'}W_{\ell' i'}] \\ &= \begin{cases} \mathbb{E}[W_{ik}^4 W_{k\ell}^2 W_{\ell i}^2] \le C\theta_i^2\theta_k^2\theta_\ell^2, & \text{if } \ell' = \ell,\ \{i',k'\} = \{i,k\}; \\ \mathbb{E}[W_{ik}^3 W_{k\ell}^2 W_{\ell i}^3] \le C\theta_i^2\theta_k^2\theta_\ell^2, & \text{if } (\ell',k') = (k,\ell),\ i' = i; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Combining it with (97) gives

$$\mathrm{Var}\Big(\frac{1}{v}\sum_{i,k,\ell(dist)} \beta_{ik\ell}W_{ik}^2 W_{k\ell}W_{\ell i}\Big) \le \frac{C\|\theta\|^4}{\|\theta\|_1^4}\sum_{i,j,k,\ell}\theta_i^2\theta_k^2\theta_\ell^2 \le \frac{C\|\theta\|^{10}}{\|\theta\|_1^4}.$$

Regarding the second term, for $W_{is}W_{kt}W_{k\ell}W_{\ell i}$ and $W_{i's'}W_{k't'}W_{k'\ell'}W_{\ell' i'}$ to be correlated, all W terms have to be perfectly matched. For each fixed (i,k,ℓ,s,t), there are only a constant number of (i',k',ℓ',s',t') so that the above is satisfied. Mimicking the argument in (70), we have

$$\begin{aligned} \mathrm{Var}\Big(\frac{1}{v}\sum_{\substack{i,k,\ell(dist) \\ s\notin\{i,\ell\},t\notin\{k,\ell\} \\ (s,t)\neq(k,i)}} \beta_{ik\ell}W_{is}W_{kt}W_{k\ell}W_{\ell i}\Big) &\le \frac{C}{v^2}\sum_{\substack{i,k,\ell(dist) \\ s\notin\{i,\ell\},t\notin\{k,\ell\} \\ (s,t)\neq(k,i)}} \beta_{ik\ell}^2 \cdot \mathrm{Var}(W_{is}W_{kt}W_{k\ell}W_{\ell i}) \\ &\le \frac{C}{\|\theta\|_1^4}\sum_{i,k,\ell,s,t}\|\theta\|^4 \cdot \theta_i^2\theta_k^2\theta_\ell^2\theta_s\theta_t \le \frac{C\|\theta\|^{10}}{\|\theta\|_1^2}. \end{aligned}$$

It follows that

$$\mathrm{Var}(Z_{1c}^{\dagger}) \leq \frac{C\|\theta\|^{10}}{\|\theta\|_1^2}.$$

Combining the above results and noticing that $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, we immediately have

$$\mathrm{Var}(Z_{1c}) \leq 3\mathrm{Var}(\widetilde{Z}_{1c}) + 3\mathrm{Var}(Z_{1c}^{*}) + 3\mathrm{Var}(Z_{1c}^{\dagger}) \leq \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}. \tag{99}$$

We now combine (95), (96), (98), and (99). Since $Z_1 = Z_{1a} + Z_{1b} + Z_{1c}$, it follows that

$$\mathbb{E}[Z_1] = \|\theta\|^4 \cdot [1 + o(1)], \qquad \mathrm{Var}(Z_1) \leq C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8).$$

This proves the claims of Z_1.

Next, we analyze Z_2. Since $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$, by direct calculations,

$$Z_2 = \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i}$$
$$+ \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j W_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} + \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j W_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i}.$$

By relabeling the indices, we find out that the first and last sums are equal and that the second and third sums are equal. It follows that

$$\begin{aligned} Z_2 = 2 &\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} \\ &+2 \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)W_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i} \\ &\equiv Z_{2a} + Z_{2b}. \end{aligned} \tag{100}$$

First, we study Z_{2a}. It is seen that

$$\begin{aligned} Z_{2a} &= 2 \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)W_{jk}\eta_k\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq \ell} W_{\ell t}\Big)W_{\ell i} \\ &= \frac{2}{v} \sum_{\substack{i,j,k,\ell(dist) \\ s\neq j, t\neq \ell}} \eta_i\eta_k W_{js}W_{jk}W_{\ell t}W_{\ell i}. \end{aligned}$$

We divide summands into four groups: (i) $s = k$ and $t = i$, (ii) $s = k$ and $t \neq i$, (iii) $s \neq k$ and $t = i$, (iv) $s \neq k$ and $t \neq i$. By symmetry between (j, k, s) and (ℓ, i, t), the sum of group (ii) and group (iii) are equal. We end up with

$$\begin{aligned} Z_{2a} &= \frac{2}{v} \sum_{i,j,k,\ell(dist)} \eta_i\eta_k W_{jk}^2 W_{\ell i}^2 + \frac{4}{v} \sum_{\substack{i,j,k,\ell(dist) \\ s\notin\{j,k\}}} \eta_i\eta_k W_{js}W_{jk}W_{\ell i}^2 \\ &\quad + \frac{2}{v} \sum_{\substack{i,j,k,\ell(dist) \\ s\notin\{j,k\}, t\notin\{\ell,i\}}} \eta_i\eta_k W_{js}W_{jk}W_{\ell t}W_{\ell i} \\ &\equiv \widetilde{Z}_{2a} + Z_{2a}^{*} + Z_{2a}^{\dagger}, \end{aligned}$$

Only $\widetilde{Z}_{2a}$ has a nonzero mean. It follows that

$$\mathbb{E}[Z_{2a}] = \mathbb{E}[\widetilde{Z}_{2a}] = \frac{2}{v} \sum_{i,j,k,\ell(dist)} \eta_i\eta_k\Omega_{jk}(1 - \Omega_{jk})\Omega_{\ell i}(1 - \Omega_{\ell i}).$$

Under the null hypothesis, $\Omega_{ij} = \theta_i\theta_j$. Hence, $\Omega_{jk}(1-\Omega_{jk})\Omega_{\ell i}(1-\Omega_{\ell i}) = \theta_j\theta_k\theta_\ell\theta_i \cdot [1 + O(\theta_{\max}^2)]$. Additionally, in the proof of (98), we have seen that $v = [1+o(1)] \cdot \|\theta\|_1^2$ and $\eta_j = [1+o(1)] \cdot \theta_j$. Combining these results gives

$$\begin{aligned}
\mathbb{E}[Z_{2a}] &= \frac{2[1+o(1)]}{\|\theta\|_1^2} \sum_{i,j,k,\ell(dist)} (\theta_i\theta_k)(\theta_j\theta_k\theta_\ell\theta_i) \\
&= \frac{2[1+o(1)]}{\|\theta\|_1^2} \Big[\sum_{i,j,k,\ell} \theta_i^2\theta_j\theta_k^2\theta_\ell - \sum_{\substack{i,j,k,\ell \\ \text{(not dist)}}} \theta_i^2\theta_j\theta_k^2\theta_\ell \Big] \\
&= \frac{2[1+o(1)]}{\|\theta\|_1^2} \Big[\|\theta\|^4\|\theta\|_1^2 - O\big(\|\theta\|_4^4\|\theta\|_1^2 + \|\theta\|_3^3\|\theta\|^2\|\theta\|_1 + \|\theta\|^6\big) \Big] \\
&= \frac{2[1+o(1)]}{\|\theta\|_1^2} \cdot \|\theta\|^4\|\theta\|_1^2[1+o(1)] \\
&= [1+o(1)] \cdot 2\|\theta\|^4. \qquad (101)
\end{aligned}$$

We then bound the variance of Z_a. Consider $\widetilde{Z}_{2a}$ first. Note that $W_{jk}^2 W_{\ell i}^2$ and $W_{j'k'}^2 W_{\ell' i'}^2$ are correlated only if either $\{j', k'\} = \{j, k\}$ or $\{j', k'\} = \{\ell, i\}$. By symmetry, it suffices to consider $\{j', k'\} = \{j, k\}$. Direct calculations show that

$$\begin{aligned}
&\mathrm{Cov}(\eta_i\eta_k W_{jk}^2 W_{\ell i}^2,\, \eta_{i'}\eta_{k'} W_{j'k'}^2 W_{\ell' i'}^2) \\
&\leq \begin{cases}
\eta_k^2\eta_i^2 \mathbb{E}[W_{jk}^4 W_{\ell i}^4] \leq C\theta_i^3\theta_j\theta_k^3\theta_\ell, & \text{if } (j',k') = (j,k),\, i = i',\, \ell = \ell'; \\
\eta_k^2\eta_i^2 \mathbb{E}[W_{jk}^4 W_{\ell i}^2 W_{\ell' i}^2] \leq C\theta_i^4\theta_j\theta_k^3\theta_\ell\theta_{\ell'}, & \text{if } (j',k') = (j,k),\, i = i',\, \ell \neq \ell'; \\
\eta_k^2\eta_i\eta_{i'} \mathbb{E}[W_{jk}^4 W_{\ell i}^2 W_{\ell' i'}^2] \leq C\theta_i^2\theta_j\theta_k^3\theta_\ell\theta_{i'}^2\theta_{\ell'}, & \text{if } (j',k') = (j,k),\, i \neq i'; \\
\eta_j\eta_k\eta_i^2 \mathbb{E}[W_{jk}^4 W_{\ell i}^4] \leq C\theta_i^3\theta_j^2\theta_k^2\theta_\ell, & \text{if } (j',k') = (k,j),\, i = i',\, \ell = \ell'; \\
\eta_j\eta_k\eta_i^2 \mathbb{E}[W_{jk}^4 W_{\ell i}^2 W_{\ell' i}^2] \leq C\theta_i^4\theta_j^2\theta_k^2\theta_\ell\theta_{\ell'}, & \text{if } (j',k') = (k,j),\, i = i',\, \ell \neq \ell'; \\
\eta_j\eta_k\eta_i\eta_{i'} \mathbb{E}[W_{jk}^4 W_{\ell i}^2 W_{\ell' i'}^2] \leq C\theta_i^2\theta_j^2\theta_k^2\theta_\ell\theta_{i'}^2\theta_{\ell'}, & \text{if } (j',k') = (k,j),\, i \neq i'; \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathrm{Var}(\widetilde{Z}_{2a}) &= \frac{4}{v^2} \sum_{\substack{i,j,k,\ell(dist) \\ i',j',k',\ell'(dist)}} \mathrm{Cov}(\eta_i\eta_k W_{jk}^2 W_{\ell i}^2,\, \eta_{i'}\eta_{k'} W_{j'k'}^2 W_{\ell' i'}^2) \\
&\leq \frac{C}{\|\theta\|_1^4} \big(\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|_4^4\|\theta\|_3^3\|\theta\|_1^3 + \|\theta\|_3^3\|\theta\|^4\|\theta\|_1^3 \\
&\qquad + \|\theta\|_3^3\|\theta\|^4\|\theta\|_1 + \|\theta\|_4^4\|\theta\|^4\|\theta\|_1^2 + \|\theta\|^8\|\theta\|_1^2 \big) \\
&\leq \frac{C\|\theta\|^4\|\theta\|_3^3}{\|\theta\|_1},
\end{aligned}$$

where the last line is obtained as follows: There are six terms in the brackets; since $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, the last three terms are dominated by the first three terms; for the first three terms, since $\|\theta\|_3^3 \leq \theta_{\max}^2\|\theta\|_1 = o(\|\theta\|_1)$ and $\|\theta\|_4^4 \leq \theta_{\max}^2\|\theta\|^2 = o(\|\theta\|^2)$, the third term dominates. Consider Z_{2a}^* next. We note that for

$$\mathbb{E}[W_{js}W_{jk}W_{\ell i}^2 \cdot W_{j's'}W_{j'k'}W_{\ell' i'}^2]$$

to be nonzero, it has to be the case of either $(W_{j's'}, W_{j'k'}) = (W_{js}, W_{jk})$ or $(W_{j's'}, W_{j'k'}) = (W_{jk}, W_{js})$. This can only happen if $(j', s', k') = (j, s, k)$ or $(j', s', k') = (j, k, s)$. By elementary calculations,

$$\eta_i\eta_k\eta_{i'}\eta_{k'} \cdot \mathbb{E}[W_{js}W_{jk}W_{\ell i}^2 \cdot W_{j's'}W_{j'k'}W_{\ell' i'}^2]$$

$$
= \begin{cases}
\eta_i^2\eta_k^2\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^4] \le C\theta_i^3\theta_j^2\theta_k^3\theta_\ell\theta_s, & \text{if } (j',s',k')=(j,s,k),\ i'=i,\ \ell'=\ell;\\
\eta_i^2\eta_k^2\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^2W_{\ell' i}^2] \le C\theta_i^4\theta_j^2\theta_k^3\theta_\ell\theta_s\theta_{\ell'}, & \text{if } (j',s',k')=(j,s,k),\ i'=i,\ \ell'\neq\ell;\\
\eta_i\eta_{i'}\eta_k^2\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^2W_{\ell' i'}^2] \le C\theta_i^2\theta_j^2\theta_k^3\theta_\ell\theta_s\theta_{i'}^2\theta_{\ell'}, & \text{if } (j',s',k')=(j,s,k),\ i\neq i';\\
\eta_i^2\eta_k\eta_s\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^4] \le C\theta_i^3\theta_j^2\theta_k^2\theta_\ell\theta_s^2, & \text{if } (j',s',k')=(j,k,s),\ i'=i,\ \ell'=\ell;\\
\eta_i^2\eta_k\eta_s\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^2W_{\ell' i}^2] \le C\theta_i^4\theta_j^2\theta_k^2\theta_\ell\theta_s^2\theta_{\ell'}, & \text{if } (j',s',k')=(j,k,s),\ i'=i,\ \ell'\neq\ell;\\
\eta_i\eta_{i'}\eta_k\eta_s\,\mathbb{E}[W_{js}^2W_{jk}^2W_{\ell i}^2W_{\ell' i'}^2] \le C\theta_i^2\theta_j^2\theta_k^2\theta_\ell\theta_s^2\theta_{i'}^2\theta_{\ell'}, & \text{if } (j',s',k')=(j,k,s),\ i\neq i';\\
0, & \text{otherwise.}
\end{cases}
$$

It follows that

$$
\begin{aligned}
\mathrm{Var}(Z_{2a}^*) &= \frac{16}{v^2}\sum_{\substack{i,j,k,\ell(dist)\\ i',j',k',\ell'(dist)}} \eta_i\eta_k\eta_{i'}\eta_{k'}\cdot\mathbb{E}[W_{js}W_{jk}W_{\ell i}^2\cdot W_{j's'}W_{j'k'}W_{\ell' i'}^2]\\
&\le \frac{C}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|^2\|\theta\|_1^2+\|\theta\|_4^4\|\theta\|_3^3\|\theta\|^2\|\theta\|_1^3+\|\theta\|_3^3\|\theta\|^6\|\theta\|_1^3\\
&\qquad+\|\theta\|_3^3\|\theta\|^6\|\theta\|_1+\|\theta\|_4^4\|\theta\|^6\|\theta\|_1^2+\|\theta\|^{10}\|\theta\|_1^2\big)\\
&\le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1},
\end{aligned}
$$

where the last inequality is obtained similarly as in the calculation of $\mathrm{Var}(\widetilde{Z}_{2a})$. Last, consider $Z_{2a}^\dagger$. Write

$$
Z_{2a}^\dagger = \frac{2}{v}\sum_{i,j,k,\ell(dist)}\eta_i\eta_kW_{j\ell}^2W_{jk}W_{\ell i} + \frac{2}{v}\sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{j,k\},t\notin\{\ell,i\}\\ (s,t)\neq(\ell,j)}}\eta_i\eta_kW_{js}W_{jk}W_{\ell t}W_{\ell i} \tag{102}
$$

Regarding the first term, we note that

$$
\begin{aligned}
&\eta_i\eta_k\eta_{i'}\eta_{k'}\cdot\mathbb{E}[W_{j\ell}^2W_{jk}W_{\ell i}\cdot W_{j'\ell'}^2W_{j'k'}W_{\ell' i'}]\\
&= \begin{cases}
\eta_i^2\eta_k^2\,\mathbb{E}[W_{jk}^2W_{\ell i}^2W_{j\ell}^4] \le C\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2, & \text{if } (j',k')=(j,k),\ (i',\ell')=(i,\ell);\\
\eta_i\eta_k^2\eta_\ell\,\mathbb{E}[W_{jk}^2W_{\ell i}^2W_{j\ell}^2W_{ji}^2] \le C\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3, & \text{if } (j',k')=(j,k),\ (i',\ell')=(\ell,i);\\
\eta_i^2\eta_k\eta_\ell\,\mathbb{E}[W_{jk}^2W_{\ell i}^2W_{j\ell}^2W_{k\ell}^2] \le C\theta_i^3\theta_j^2\theta_k^3\theta_\ell^4, & \text{if } (j',k')=(k,j),\ (i',\ell')=(i,\ell);\\
\eta_i\eta_k\eta_\ell\eta_j\,\mathbb{E}[W_{jk}^2W_{\ell i}^2W_{j\ell}^2W_{ki}^2] \le C\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3, & \text{if } (j',k')=(k,j),\ (i',\ell')=(\ell,i);\\
0, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathrm{Var}\Big(\frac{2}{v}\sum_{i,j,k,\ell(dist)}\eta_i\eta_kW_{j\ell}^2W_{jk}W_{\ell i}\Big)\\
\le&\frac{C}{\|\theta\|_1^4}\sum_{i,j,k,\ell}(\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2+\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3+\theta_i^3\theta_j^2\theta_k^3\theta_\ell^4)\\
\le&\frac{C}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|^4+\|\theta\|_3^{12}+\|\theta\|_4^4\|\theta\|_3^6\|\theta\|^2\big)\\
\le&\frac{C\|\theta\|_3^6\|\theta\|^4}{\|\theta\|_1^4}.
\end{aligned}
$$

Regarding the second term in (102). We note that, for $\eta_i\eta_k W_{js}W_{jk}W_{\ell t}W_{\ell i}$ and $\eta_{i'}\eta_{k'}W_{j's'}W_{j'k'}W_{\ell' t'}W_{\ell' i'}$ to be correlated, all the W terms have to be perfectly paired. It turns out that

$$\mathbb{E}[W_{js}W_{jk}W_{\ell t}W_{\ell i}\cdot W_{j's'}W_{j'k'}W_{\ell't'}W_{\ell'i'}] = \mathbb{E}[W_{js}^2 W_{jk}^2 W_{\ell t}^2 W_{\ell i}^2].$$

To perfectly pair the W terms, there are two possible cases: (i) $(j',\ell') = (j,\ell)$, $\{s',k'\} = \{s,k\}$, $\{\ell',i'\} = \{\ell,i\}$. (ii) $(j',\ell') = (\ell,j)$, $\{s',k'\} = \{\ell,i\}$, $\{\ell',i'\} = \{s,k\}$. As a result, $\eta_i\eta_k\eta_{i'}\eta_{k'}$ only has the following possibilities:

$$\begin{aligned}&\eta_i\eta_k(\eta_i\eta_k) = \eta_i^2\eta_k^2,\ \eta_i\eta_k(\eta_i\eta_s) = \eta_i^2\eta_k\eta_s,\ \eta_i\eta_k(\eta_\ell\eta_k) = \eta_i\eta_k^2\eta_\ell,\ \eta_i\eta_k(\eta_\ell\eta_s) = \eta_i\eta_k\eta_\ell\eta_s,\\ &\eta_i\eta_k(\eta_k\eta_i) = \eta_i^2\eta_k^2,\ \eta_i\eta_k(\eta_k\eta_\ell) = \eta_i\eta_k^2\eta_\ell,\ \eta_i\eta_k(\eta_s\eta_i) = \eta_i^2\eta_k\eta_s,\ \eta_i\eta_k(\eta_s\eta_\ell) = \eta_i\eta_k\eta_\ell\eta_s.\end{aligned}$$

By symmetry, there are only three different types: $\eta_i^2\eta_k^2$, $\eta_i^2\eta_k\eta_s$, and $\eta_i\eta_k\eta_\ell\eta_s$. It follows that

$$\begin{aligned}&\mathrm{Var}\Big(\frac{2}{v}\sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{j,k\},t\notin\{\ell,i\},(s,t)\neq(\ell,j)}}\eta_i\eta_k W_{js}W_{jk}W_{\ell t}W_{\ell i}\Big)\\ &\leq\frac{C}{\|\theta\|_1^4}\sum_{i,j,k,\ell,s,t}(\theta_i^2\theta_k^2+\theta_i^2\theta_k\theta_s+\theta_i\theta_k\theta_\ell\theta_s)\cdot\theta_j^2\theta_s\theta_k\theta_\ell^2\theta_t\theta_i\\ &\leq\frac{C}{\|\theta\|_1^4}\sum_{i,j,k,\ell,s,t}(\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2\theta_s\theta_t+\theta_i^3\theta_j^2\theta_k^2\theta_\ell^2\theta_s^2\theta_t+\theta_i^2\theta_j^2\theta_k^2\theta_\ell^3\theta_s^2\theta_t)\\ &\leq\frac{C}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|^4\|\theta\|_1^2+\|\theta\|_3^3\|\theta\|^8\|\theta\|_1\big)\ \leq\frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2}.\end{aligned}$$

It follows that

$$\mathrm{Var}(Z_{2a}^{\dagger})\leq\frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2}.$$

Comparing the variances of $\widetilde{Z}_{2a}$, Z_{2a}^* and $Z_{2a}^{\dagger}$, we find out that the variance of Z_{2a}^* dominates. As a result,

$$\mathrm{Var}(Z_{2a})\leq 3\mathrm{Var}(\widetilde{Z}_{2a})+3\mathrm{Var}(Z_{2a}^*)+3\mathrm{Var}(Z_{2a}^{\dagger})\leq\frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}. \tag{103}$$

Second, we study Z_{2b}. It is seen that

$$\begin{aligned}Z_{2b} &= 2\sum_{i,j,k,\ell(dist)}\eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j}W_{js}\Big)W_{jk}\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq k}W_{kt}\Big)\eta_\ell W_{\ell i}\\ &= \frac{2}{v}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq j,t\neq k}}\eta_i\eta_\ell W_{js}W_{jk}W_{kt}W_{\ell i}.\end{aligned}$$

We divide summands into four groups: (i) $s=k$ and $t=j$, (ii) $s=k$ and $t\neq j$, (iii) $s\neq k$ and $t=j$, (iv) $s\neq k$ and $t\neq j$. By index symmetry, the sums of group (ii) and group (iii) are equal. We end up with

$$\begin{aligned}Z_{2b} &= \frac{2}{v}\sum_{i,j,k,\ell(dist)}\eta_i\eta_\ell W_{jk}^3W_{\ell i}+\frac{4}{v}\sum_{i,j,k,\ell(dist),t\notin\{k,j\}}\eta_i\eta_\ell W_{jk}^2W_{kt}W_{\ell i}\\ &\quad+\frac{2}{v}\sum_{i,j,k,\ell(dist),s\neq\{j,k\},t\neq\{j,k\}}\eta_i\eta_\ell W_{js}W_{jk}W_{kt}W_{\ell i}\\ &\equiv\widetilde{Z}_{2b}+Z_{2b}^*+Z_{2b}^{\dagger}.\end{aligned}$$

It is easy to see that all three terms have mean zero. Therefore,

$$\mathbb{E}[Z_{2b}] = 0. \tag{104}$$

We then bound the variances. Consider $\widetilde{Z}_{2b}$ first. By direct calculations,

$$\begin{aligned}
&\eta_i\eta_\ell\eta_{i'}\eta_{\ell'}\cdot\mathbb{E}[W_{jk}^3W_{\ell i}\cdot W_{j'k'}^3W_{\ell' i'}]\\
&=\begin{cases}\eta_i^2\eta_\ell^2\cdot\mathbb{E}[W_{jk}^6W_{\ell i}^2]\leq C\theta_i^3\theta_j\theta_k\theta_\ell^3, & \text{if } \{j',k'\}=\{j,k\},\ \{\ell',i'\}=\{\ell,i\};\\ \eta_i\eta_\ell\eta_j\eta_k\cdot\mathbb{E}[W_{jk}^4W_{\ell i}^4]\leq C\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2, & \text{if } \{j',k'\}=\{\ell,i\},\ \{\ell',i'\}=\{j,k\};\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Var}(\widetilde{Z}_{2b}) &\leq \frac{C}{\|\theta\|_1^4}\Big(\sum_{i,j,k,\ell}\theta_i^3\theta_j\theta_k\theta_\ell^3+\sum_{i,j,k,\ell}\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\Big)\\
&\leq \frac{C}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|_1^2+\|\theta\|^8\big)\\
&\leq \frac{C\|\theta\|_3^6}{\|\theta\|_1^2}.
\end{aligned}$$

Consider Z_{2b}^* next. By direct calculations,

$$\begin{aligned}
&\eta_i\eta_\ell\eta_{i'}\eta_{\ell'}\cdot\mathbb{E}[W_{jk}^2W_{kt}W_{\ell i}\cdot W_{j'k'}^2W_{k't'}W_{\ell' i'}]\\
&=\begin{cases}\eta_i^2\eta_\ell^2\,\mathbb{E}[W_{jk}^4W_{kt}^2W_{\ell i}^2]\leq C\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t, & \text{if } (k',t')=(k,t),\ \{\ell',i'\}=\{\ell,i\},\ j'=j;\\
\eta_i^2\eta_\ell^2\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^2W_{j'k}^2]\leq C\theta_i^3\theta_j\theta_k^3\theta_\ell^3\theta_t\theta_{j'}, & \text{if } (k',t')=(k,t),\ \{\ell',i'\}=\{\ell,i\},\ j'\neq j;\\
\eta_i^2\eta_\ell^2\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^2W_{j't}^2]\leq C\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t^2\theta_{j'}, & \text{if } (k',t')=(t,k),\ \{\ell',i'\}=\{\ell,i\};\\
\eta_i\eta_\ell\eta_k\eta_t\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^4]\leq C\theta_i^2\theta_j\theta_k^3\theta_\ell^2\theta_t^2, & \text{if } (k',t')=(\ell,i),\ \{\ell',i'\}=\{k,t\},\ j'=i;\\
\eta_i\eta_\ell\eta_k\eta_t\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^2W_{j'\ell}^2]\leq C\theta_i^2\theta_j\theta_k^3\theta_\ell^3\theta_t^2\theta_{j'}, & \text{if } (k',t')=(\ell,i),\ \{\ell',i'\}=\{k,t\},\ j'\neq i;\\
\eta_i\eta_\ell\eta_k\eta_t\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^4]\leq C\theta_i^2\theta_j\theta_k^3\theta_\ell^2\theta_t^2, & \text{if } (k',t')=(i,\ell),\ \{\ell',i'\}=\{k,t\},\ j'=\ell;\\
\eta_i\eta_\ell\eta_k\eta_t\,\mathbb{E}[W_{jk}^2W_{kt}^2W_{\ell i}^2W_{j'i}^2]\leq C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_t^2\theta_{j'}, & \text{if } (k',t')=(i,\ell),\ \{\ell',i'\}=\{k,t\},\ j'\neq \ell;\\
\eta_i^2\eta_\ell^2\,\mathbb{E}[W_{jk}^3W_{kt}^3W_{\ell i}^2]\leq C\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t, & \text{if } (k',t',j')=(k,j,t),\ \{i',\ell'\}=\{i,\ell\};\\
0, & \text{otherwise.}\end{cases}
\end{aligned}$$

There are only two four types on the right hand side. It follows that

$$\begin{aligned}
\mathrm{Var}(Z_{2b}^*) &\leq \frac{C}{\|\theta\|_1^4}\Big(\sum_{i,j,k,\ell,t,j'}\theta_i^3\theta_j\theta_k^3\theta_\ell^3\theta_t\theta_{j'}+\sum_{i,j,k,\ell,t,j'}\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t^2\theta_{j'}\\
&\qquad+\sum_{i,j,k,\ell,t}\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t+\sum_{i,j,k,\ell,t}\theta_i^2\theta_j\theta_k^3\theta_\ell^2\theta_t^2\Big)\\
&\leq \frac{C}{\|\theta\|_1^4}\big(\|\theta\|_3^9\|\theta\|_1^3+\|\theta\|_3^6\|\theta\|^4\|\theta\|_1^2+\|\theta\|_3^6\|\theta\|^2\|\theta\|_1^2+\|\theta\|_3^3\|\theta\|^6\|\theta\|_1\big)\\
&\leq \frac{C\|\theta\|_3^9}{\|\theta\|_1}.
\end{aligned}$$

Last, consider $Z_{2b}^\dagger$. By direct calculations,

$$\eta_i\eta_\ell\eta_{i'}\eta_{\ell'}\cdot\mathbb{E}[W_{js}W_{jk}W_{kt}W_{\ell i}\cdot W_{j's'}W_{j'k'}W_{k't'}W_{\ell' i'}]$$

$$= \begin{cases} \eta_i^2\eta_\ell^2\,\mathbb{E}[W_{js}^2W_{jk}^2W_{kt}^2W_{\ell i}^2] \le C\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_s\theta_t, & \text{if } (j',s')=(j,s),\ (k',t')=(k,t),\ \{\ell',i'\}=\{\ell,i\}; \\ \eta_i^2\eta_\ell^2\,\mathbb{E}[W_{js}^2W_{jk}^2W_{kt}^2W_{\ell i}^2] \le C\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_s\theta_t, & \text{if } (j',s')=(k,t),\ (k',t')=(j,s),\ \{\ell',i'\}=\{\ell,i\}; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{Var}(Z_{2b}^\dagger) \le \frac{C}{\|\theta\|_1^4}\sum_{i,j,k,\ell,s,t}\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_s\theta_t \le \frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2}.$$

Since $\|\theta\|_1\|\theta\|_3^3 \ge \|\theta\|^4 \to \infty$, the variance of Z_{2b}^* dominates the variances of $\widetilde{Z}_{2b}$ and $Z_{2b}^\dagger$. We thus have

$$\mathrm{Var}(Z_{2b}) \le 3\mathrm{Var}(\widetilde{Z}_{2b}) + 3\mathrm{Var}(Z_{2b}^*) + 3\mathrm{Var}(Z_{2b}^\dagger) \le \frac{C\|\theta\|_3^9}{\|\theta\|_1}. \tag{105}$$

We now combine (101), (103), (104), and (105). Since $\|\theta\|_3^6 \le \theta_{\max}^2\|\theta\|^4 \ll \|\theta\|^6$, the right hand side of (105) is much smaller than the right hand side of (103). It yields that

$$\mathbb{E}[Z_2] = 2\|\theta\|^4\cdot[1+o(1)], \qquad \mathrm{Var}(Z_2) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8).$$

This proves the claims of Z_2.

G.4.7. *Proof of Lemma G.7.* It suffices to prove the claims for each of Z_1-Z_6. We have analyzed Z_1-Z_2 under the null hypothesis. The proof for the alternative hypothesis is similar and omitted. We obtain that

$$\big|\mathbb{E}[Z_1]\big| \le C\|\theta\|^4, \qquad \mathrm{Var}(Z_1) \le C\|\theta\|^2\|\theta\|_3^6 = o(\|\theta\|^8),$$
$$\big|\mathbb{E}[Z_2]\big| \le C\|\theta\|^4, \qquad \mathrm{Var}(Z_2) \le \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8).$$

First, we analyze Z_3. Since $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$, we have

$$Z_3 = \sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j-\tilde{\eta}_j)\eta_j(\eta_k-\tilde{\eta}_k)\widetilde{\Omega}_{k\ell}W_{\ell i} + \sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j-\tilde{\eta}_j)^2\eta_k\widetilde{\Omega}_{k\ell}W_{\ell i}$$
$$+ \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i-\tilde{\eta}_i)\eta_j^2(\eta_k-\tilde{\eta}_k)\widetilde{\Omega}_{k\ell}W_{\ell i} + \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i-\tilde{\eta}_i)\eta_j(\eta_j-\tilde{\eta}_j)\eta_k\widetilde{\Omega}_{k\ell}W_{\ell i}$$
$$\equiv Z_{3a} + Z_{3b} + Z_{3c} + Z_{3d}. \tag{106}$$

First, we study Z_{3a}. By direct calculations,

$$Z_{3a} = \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j}W_{js}\Big)\eta_j\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq k}W_{kt}\Big)\widetilde{\Omega}_{k\ell}W_{\ell i}$$
$$= \frac{1}{v}\sum_{\substack{i,j,k,\ell(dist) \\ s\neq j, t\neq k}} \beta_{ijk\ell}W_{js}W_{kt}W_{\ell i}, \qquad \text{where } \beta_{ijk\ell} = \eta_i\eta_j\widetilde{\Omega}_{k\ell}.$$

Since (i,j,k,ℓ) are distinct, all summands have mean zero. Hence,

$$\mathbb{E}[Z_{3a}] = 0. \tag{107}$$

To bound its variance, re-write

$$Z_{3a} = \frac{1}{v}\sum_{i,j,k,\ell(dist)} \beta_{ijk\ell} W_{jk}^2 W_{\ell i} + \frac{1}{v}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq j, t\neq k, (s,t)\neq(k,j)}} \beta_{ijk\ell} W_{js} W_{kt} W_{\ell i}$$
$$\equiv \widetilde{Z}_{3a} + Z_{3a}^*.$$

We note that $|\beta_{ijk\ell}| \leq C\alpha\theta_i\theta_j\theta_k\theta_\ell$ by (74) and (81). Consider the variance of $\widetilde{Z}_{3a}$. By direct calculations,

$$\beta_{ijk\ell}\beta_{i'j'k'\ell'} \cdot \mathrm{Cov}(W_{jk}^2 W_{\ell i}, W_{j'k'}^2 W_{\ell' i'})$$
$$= \begin{cases} C\alpha^2\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\, \mathbb{E}[W_{jk}^4 W_{\ell i}^2] \leq C\alpha^2\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3, & \text{if } \{\ell', i'\} = \{\ell, i\}, \{j', k'\} = \{j, k\};\\ C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2\theta_{j'}\theta_{k'}\, \mathbb{E}[W_{jk}^2 W_{j'k'}^2 W_{\ell i}^2] \leq C\alpha^2\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_{j'}^2\theta_{k'}^2, & \text{if } \{\ell', i'\} = \{\ell, i\}, \{j', k'\} \neq \{j, k\};\\ C\alpha^2\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\, \mathbb{E}[W_{jk}^3 W_{\ell i}^3] \leq C\alpha^2\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3, & \text{if } \{j', k'\} = \{\ell, i\}, \{\ell', i'\} = \{j, k\};\\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{Var}(\widetilde{Z}_{3a}) \leq \frac{C\alpha^2}{\|\theta\|_1^4}\Big(\sum_{i,j,k,\ell} \theta_i^3\theta_j^3\theta_k^3\theta_\ell^3 + \sum_{i,j,k,\ell,j',k'} \theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_{j'}^2\theta_{k'}^2\Big)$$
$$\leq \frac{C\alpha^2}{\|\theta\|_1^4}\big(\|\theta\|_3^{12} + \|\theta\|^8\|\theta\|_3^6\big)$$
$$\leq \frac{C\alpha^2\|\theta\|_3^{12}}{\|\theta\|_1^2}.$$

Consider the variance of Z_{3a}^*. For $W_{js}W_{kt}W_{\ell i}$ and $W_{j's'}W_{k't'}W_{\ell' i'}$ to be correlated, all W terms have to be perfectly paired. By symmetry across indices, it reduces to three cases: (i) $(\ell', i') = (\ell, i)$, $(j', s') = (j, s)$, $(k', t') = (k, t)$; (ii) $(\ell', i') = (j, s)$, $(j', s') = (\ell, i)$, $(k', t') = (k, t)$; (iii) $(\ell', i') = (j, s)$, $(j', s') = (k, t)$, $(k', t') = (\ell, i)$. It follows that

$$\beta_{ijk\ell}\beta_{i'j'k'\ell'} \cdot \mathbb{E}[W_{js}W_{kt}W_{\ell i} \cdot W_{j's'}W_{k't'}W_{\ell' i'}]$$
$$\leq C\alpha^2(\theta_i\theta_j\theta_k\theta_\ell)(\theta_{i'}\theta_{j'}\theta_{k'}\theta_{\ell'}) \cdot \mathbb{E}[W_{js}^2 W_{kt}^2 W_{\ell i}^2]$$
$$\leq \begin{cases} C\alpha^2\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\mathbb{E}[W_{js}^2 W_{kt}^2 W_{\ell i}^2] \leq C\alpha^2\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3\theta_s\theta_t, & \text{case (i)}\\ C\alpha^2(\theta_i\theta_j\theta_k\theta_\ell)(\theta_s\theta_\ell\theta_k\theta_j)\mathbb{E}[W_{js}^2 W_{kt}^2 W_{\ell i}^2] \leq C\alpha^2\theta_i^2\theta_j^3\theta_k^3\theta_\ell^3\theta_s^2\theta_t, & \text{case (ii)}\\ C\alpha^2(\theta_i\theta_j\theta_k\theta_\ell)(\theta_s\theta_k\theta_\ell\theta_j)\mathbb{E}[W_{js}^2 W_{kt}^2 W_{\ell i}^2] \leq C\alpha^2\theta_i^2\theta_j^3\theta_k^3\theta_\ell^3\theta_s^2\theta_t, & \text{case (iii)}\\ 0, & \text{otherwise.} \end{cases}$$

As a result,

$$\mathrm{Var}(Z_{3a}^*) \leq \frac{C}{\|\theta\|_1^4}\Big(\sum_{i,j,k,\ell,s,t} \alpha^2\theta_i^3\theta_j^3\theta_k^3\theta_\ell^3\theta_s\theta_t + \sum_{i,j,k,\ell,s,t} \alpha^2\theta_i^2\theta_j^3\theta_k^3\theta_\ell^3\theta_s^2\theta_t\Big)$$
$$\leq \frac{C\alpha^2}{\|\theta\|_1^4}\big(\|\theta\|_3^{12}\|\theta\|_1^2 + \|\theta\|^4\|\theta\|_3^9\|\theta\|_1\big)$$
$$\leq \frac{C\alpha^2\|\theta\|_3^{12}}{\|\theta\|_1^2}.$$

Combining the variance of $\widetilde{Z}_{3a}$ and Z_{3a}^* gives

$$\mathrm{Var}(Z_{3a}) \le \frac{C\alpha^2\|\theta\|_3^{12}}{\|\theta\|_1^2}. \tag{108}$$

Second, we study Z_{3b}. It is seen that

$$\begin{aligned} Z_{3b} &= \sum_{i,j,k,\ell(dist)} \eta_i\Big(-\frac{1}{\sqrt{v}}\sum_{s\neq j} W_{js}\Big)\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq j} W_{jt}\Big)\eta_k\widetilde{\Omega}_{k\ell}W_{\ell i} \\ &= \frac{1}{v}\sum_{\substack{i,j,\ell(dist)\\ s\neq j, t\neq j}}\Big(\sum_{k\notin\{i,j,\ell\}}\eta_i\eta_k\widetilde{\Omega}_{k\ell}\Big)W_{js}W_{jt}W_{\ell i} \\ &\equiv \frac{1}{v}\sum_{\substack{i,j,\ell(dist)\\ s\neq j, t\neq j}}\beta_{ij\ell}W_{js}W_{jt}W_{\ell i}, \end{aligned}$$

where by (74) and (81),

$$|\beta_{ij\ell}| \le \sum_{k\notin\{i,j,\ell\}} |\eta_i\eta_k\widetilde{\Omega}_{k\ell}| \le \sum_k C\alpha\theta_i\theta_k^2\theta_\ell \le C\alpha\|\theta\|^2\cdot\theta_i\theta_\ell. \tag{109}$$

We further decompose Z_{3b} into

$$Z_{3b} = \frac{1}{v}\sum_{\substack{i,j,\ell(dist)\\ s\neq j}}\beta_{ij\ell}W_{js}^2W_{\ell i} + \frac{1}{v}\sum_{\substack{i,j,\ell(dist)\\ s,t(dist)\notin\{j\}}}\beta_{ij\ell}W_{js}W_{jt}W_{\ell i} \equiv \widetilde{Z}_{3b} + Z_{3b}^*.$$

It is easy to see that both terms have mean zero. It follows that

$$\mathbb{E}[Z_{3b}] = 0. \tag{110}$$

To calculate the variance of $\widetilde{Z}_{3b}$, we note that

$$\begin{aligned} &\beta_{ij\ell}\beta_{i'j'\ell'}\cdot\mathbb{E}[W_{js}^2W_{\ell i}\cdot W_{j's'}^2W_{\ell'i'}] \\ &\le C\alpha^2\|\theta\|^4\theta_i\theta_{i'}\theta_\ell\theta_{\ell'}\cdot\mathbb{E}[W_{js}^2W_{\ell i}\cdot W_{j's'}^2W_{\ell'i'}] \\ &\le \begin{cases} C\alpha^2\|\theta\|^4\theta_i^2\theta_\ell^2\cdot\mathbb{E}[W_{js}^4W_{\ell i}^2] \le C\alpha^2\|\theta\|^4\theta_i^3\theta_j\theta_\ell^3\theta_s & \text{if } \{\ell',i'\}=\{\ell,i\},\ \{j',s'\}=\{j,s\} \\ C\alpha^2\|\theta\|^4\theta_i^2\theta_\ell^2\cdot\mathbb{E}[W_{js}^2W_{\ell i}^2W_{j's'}^2] \le C\alpha^2\|\theta\|^4\theta_i^3\theta_j\theta_\ell^3\theta_s\theta_{j'}\theta_{s'}, & \text{if } \{\ell',i'\}=\{\ell,i\},\ \{j',s'\}\neq\{j,s\}; \\ C\alpha^2\|\theta\|^4\theta_i\theta_\ell\theta_j\theta_s\cdot\mathbb{E}[W_{js}^3W_{\ell i}^3] \le C\alpha^2\|\theta\|^4\theta_i^2\theta_j^2\theta_\ell^2\theta_s^2, & \text{if } \{\ell',i'\}=\{j,s\},\ \{j',s'\}=\{\ell,i\}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{Var}(\widetilde{Z}_{3b}) &\le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\Big(\sum_{i,j,\ell,s}\theta_i^3\theta_j\theta_\ell^3\theta_s + \sum_{i,j,\ell,s,j',s'}\theta_i^3\theta_j\theta_\ell^3\theta_s\theta_{j'}\theta_{s'} + \sum_{i,j,\ell,s,j',s'}\theta_i^2\theta_j^2\theta_\ell^2\theta_s^2\Big) \\ &\le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|_3^6\|\theta\|_1^4 + \|\theta\|^8\big) \\ &\le C\alpha^2\|\theta\|^4\|\theta\|_3^6. \end{aligned}$$

To calculate the variance of Z_{3b}^*, we note that $\mathbb{E}[W_{js}W_{jt}W_{\ell i}\cdot W_{j's'}W_{j't'}W_{\ell'i'}]$ is nonzero only if $j'=j$, $\{s',t'\}=\{s,t\}$ and $\{\ell',i'\}=\{\ell,i\}$. Combining it with (112) gives

$$\mathrm{Var}(Z_{3b}^*) \le \frac{C}{v^2}\sum_{\substack{i,j,\ell(dist)\\ s,t(dist)\notin\{j\}}}\beta_{ij\ell}^2\cdot\mathbb{E}[W_{js}^2W_{jt}^2W_{\ell i}^2]$$

$$\leq \frac{C}{\|\theta\|_1^4} \sum_{i,j,\ell,s,t} (\alpha\|\theta\|^2\theta_i\theta_\ell)^2 \cdot \theta_j^2\theta_s\theta_t\theta_\ell\theta_i$$
$$\leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4} \sum_{i,j,\ell,s,t} \theta_i^3\theta_j^2\theta_\ell^3\theta_s\theta_t$$
$$\leq \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^6}{\|\theta\|_1^2}.$$

Since $\|\theta\|^6 \leq \|\theta\|^4\|\theta\|^2 \ll \|\theta\|^4\|\theta\|_1$, the variance of $\widetilde{Z}_{3b}$ dominates the variance of Z_{3b}^*. Combining the above gives

$$\text{Var}(Z_{3b}) \leq 2\text{Var}(\widetilde{Z}_{3b}) + 2\text{Var}(Z_{3b}^*) \leq C\alpha^2\|\theta\|^4\|\theta\|_3^6. \tag{111}$$

Third, we study Z_{3c}. It is seen that

$$Z_{3c} = \sum_{i,j,k,\ell(dist)} \Big(-\frac{1}{\sqrt{v}}\sum_{s\neq i} W_{is}\Big)\eta_j^2\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq k} W_{kt}\Big)\widetilde{\Omega}_{k\ell}W_{\ell i}$$
$$= \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\neq i, t\neq k}} \Big(\sum_{j\notin\{i,k,\ell\}} \eta_j^2\widetilde{\Omega}_{k\ell}\Big)W_{is}W_{kt}W_{\ell i}$$
$$\equiv \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\neq i, t\neq k}} \beta_{ik\ell}W_{is}W_{kt}W_{\ell i},$$

where by (74) and (81),

$$|\beta_{ik\ell}| \leq \sum_{j\notin\{i,k,\ell\}} |\eta_j^2\widetilde{\Omega}_{k\ell}| \leq \sum_j C\alpha\theta_j^2\theta_k\theta_\ell \leq C\alpha\|\theta\|^2\theta_k\theta_\ell. \tag{112}$$

There are two cases for the indices: $i=\ell$ and $i\neq\ell$. We further decompose Z_{3c} into

$$Z_{3c} = \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ t\neq k}} \beta_{ik\ell}W_{i\ell}^2W_{kt} + \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\}, t\neq k}} \beta_{ik\ell}W_{is}W_{kt}W_{\ell i} \equiv \widetilde{Z}_{3c} + Z_{3c}^*.$$

It is easy to see that both terms have zero mean. Hence,

$$\mathbb{E}[Z_{3c}] = 0. \tag{113}$$

To calculate the variance of $\widetilde{Z}_{3c}$, we note that $W_{i\ell}^2W_{kt}$ and $W_{i'\ell'}^2W_{k't'}$ are correlated only when (i) $\{k',t'\}=\{k,t\}$ or (ii) $\{k',t'\}=\{i,\ell\}$ and $\{i',\ell'\}=\{k,t\}$. By direct calculations,

$$\beta_{ik\ell}\beta_{i'k'\ell'} \cdot \mathbb{E}[W_{i\ell}^2W_{kt} \cdot W_{i'\ell'}^2W_{k't'}]$$
$$\leq C\alpha^2\|\theta\|^4\theta_k\theta_{k'}\theta_\ell\theta_{\ell'} \cdot \mathbb{E}[W_{i\ell}^2W_{kt} \cdot W_{i'\ell'}^2W_{k't'}]$$

$$
\leq \begin{cases} C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell^2\,\mathbb{E}[W_{i\ell}^4W_{kt}^2] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^3\theta_\ell^3\theta_t, & \text{if } (k',t')=(k,t),\ (i',\ell')=(i,\ell);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell\theta_i\,\mathbb{E}[W_{i\ell}^4W_{kt}^2] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^2\theta_t, & \text{if } (k',t')=(k,t),\ (i',\ell')=(\ell,i);\\ C\alpha^2\|\theta\|^4\theta_k\theta_\ell^2\theta_t\,\mathbb{E}[W_{i\ell}^4W_{kt}^2] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^2\theta_\ell^3\theta_t^2, & \text{if } (k',t')=(t,k),\ (i',\ell')=(i,\ell);\\ C\alpha^2\|\theta\|^4\theta_k\theta_t\theta_\ell\theta_i\,\mathbb{E}[W_{i\ell}^4W_{kt}^2] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^2\theta_\ell^2\theta_t^2, & \text{if } (k',t')=(t,k),\ (i',\ell')=(\ell,i);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell\theta_{\ell'}\,\mathbb{E}[W_{i\ell}^2W_{kt}^2W_{i'\ell'}^2] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^3\theta_\ell^2\theta_t\theta_{i'}\theta_{\ell'}^2, & \text{if } (k',t')=(k,t),\ \{i',\ell'\}\neq\{i,\ell\};\\ C\alpha^2\|\theta\|^4\theta_k\theta_t\theta_\ell\theta_{\ell'}\,\mathbb{E}[W_{i\ell}^2W_{kt}^2W_{i'\ell'}^2] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^2\theta_\ell^2\theta_t^2\theta_{i'}\theta_{\ell'}^2, & \text{if } (k',t')=(t,k),\ \{i',\ell'\}\neq\{i,\ell\};\\ C\alpha^2\|\theta\|^4\theta_k\theta_i\theta_\ell\theta_t\,\mathbb{E}[W_{i\ell}^3W_{kt}^3] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^2\theta_\ell^2\theta_t^2, & \text{if } (k',t')=(i,\ell),\ (i',\ell')=(k,t);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_i\theta_\ell\,\mathbb{E}[W_{i\ell}^3W_{kt}^3] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^2\theta_t, & \text{if } (k',t')=(i,\ell),\ (i',\ell')=(t,k);\\ C\alpha^2\|\theta\|^4\theta_k\theta_\ell^2\theta_t\,\mathbb{E}[W_{i\ell}^3W_{kt}^3] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^2\theta_\ell^3\theta_t^2, & \text{if } (k',t')=(\ell,i),\ (i',\ell')=(k,t);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell^2\,\mathbb{E}[W_{i\ell}^3W_{kt}^3] \leq C\alpha^2\|\theta\|^4\theta_i\theta_k^3\theta_\ell^3\theta_t, & \text{if } (k',t')=(\ell,i),\ (i',\ell')=(t,k);\\ 0, & \text{otherwise.} \end{cases}
$$

There are only five types on the right hand side. It follows that

$$
\begin{aligned}
\mathrm{Var}(\widetilde{Z}_{3c}) &\leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\Big(\sum_{i,k,\ell,t}\theta_i\theta_k^3\theta_\ell^3\theta_t + \sum_{i,k,\ell,t}\theta_i^2\theta_k^3\theta_\ell^2\theta_t + \sum_{i,k,\ell,t}\theta_i^2\theta_k^2\theta_\ell^2\theta_t^2 \\
&\qquad + \sum_{i,k,\ell,t,i',\ell'}\theta_i\theta_k^3\theta_\ell^2\theta_t\theta_{i'}\theta_{\ell'}^2 + \sum_{i,k,\ell,t,i',\ell'}\theta_i\theta_k^2\theta_\ell^2\theta_t^2\theta_{i'}\theta_{\ell'}^2\Big)\\
&\leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^4\|\theta\|_3^3\|\theta\|_1 + \|\theta\|^8 + \|\theta\|^4\|\theta\|_3^3\|\theta\|_1^3 + \|\theta\|^8\|\theta\|_1^2\big)\\
&\leq \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1},
\end{aligned}
$$

where the last inequality is obtained as follows: Among the five terms in the brackets, the first and third terms are dominated by the last term, and the second term is dominated by the fourth term; it remains to compare the fourth term and the last term, where the fourth term dominated because $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$. To calculate the variance of Z_{3c}^*, we write

$$
Z_{3c}^* = \frac{1}{v}\sum_{i,k,\ell(dist)}\beta_{ik\ell}W_{ik}^2W_{\ell i} + \frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\},t\neq k,(s,t)\neq(k,i)}}\beta_{ik\ell}W_{is}W_{kt}W_{\ell i}.
$$

Regarding the first term, we note that

$$
\begin{aligned}
&\beta_{ik\ell}\beta_{i'k'\ell'}\cdot\mathbb{E}[W_{ik}^2W_{\ell i}\cdot W_{i'k'}^2W_{\ell'i'}]\\
\leq& C\alpha^2\|\theta\|^4\theta_k\theta_\ell\theta_{k'}\theta_{\ell'}\cdot\mathbb{E}[W_{ik}^2W_{\ell i}\cdot W_{i'k'}^2W_{\ell'i'}]\\
\leq& \begin{cases} C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell^2\,\mathbb{E}[W_{ik}^4W_{\ell i}^2] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^3, & \text{if } (\ell',i')=(\ell,i),\ k'=k;\\ C\alpha^2\|\theta\|^4\theta_k\theta_\ell^2\theta_{k'}\,\mathbb{E}[W_{ik}^2W_{\ell i}^2W_{ik'}^2] \leq C\alpha^2\|\theta\|^4\theta_i^3\theta_k^2\theta_\ell^3\theta_{k'}^2, & \text{if } (\ell',i')=(\ell,i),\ k'\neq k;\\ C\alpha^2\|\theta\|^4\theta_i\theta_k\theta_\ell\theta_{k'}\,\mathbb{E}[W_{ik}^2W_{\ell i}^2W_{\ell k'}^2] \leq C\alpha^2\|\theta\|^4\theta_i^3\theta_k^2\theta_\ell^3\theta_{k'}^2, & \text{if } (\ell',i')=(i,\ell);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell^2\,\mathbb{E}[W_{ik}^3W_{\ell i}^3] \leq C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^3, & \text{if } (\ell',i')=(k,i),\ k'=\ell;\\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

It follows that

$$
\mathrm{Var}\Big(\frac{1}{v}\sum_{i,k,\ell(dist)}\beta_{ik\ell}W_{ik}^2W_{\ell i}\Big) \leq \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\Big(\sum_{i,k,\ell}\theta_i^2\theta_k^3\theta_\ell^3 + \sum_{i,k,\ell,k'}\theta_i^3\theta_k^2\theta_\ell^3\theta_{k'}^2\Big)
$$

$$\le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|^2\|\theta\|_3^6 + \|\theta\|^4\|\theta\|_3^6\big) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^4}.$$

Regarding the second term, we note that

$$\beta_{ik\ell}\beta_{i'k'\ell'}\cdot\mathbb{E}[W_{is}W_{kt}W_{\ell i}\cdot W_{i's'}W_{k't'}W_{\ell' i'}]$$

$$\le C\alpha^2\|\theta\|^4\theta_k\theta_{k'}\theta_\ell\theta_{\ell'}\cdot\mathbb{E}[W_{is}W_{kt}W_{\ell i}\cdot W_{i's'}W_{k't'}W_{\ell' i'}]$$

$$\le \begin{cases} C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell^2\,\mathbb{E}[W_{is}^2W_{kt}^2W_{\ell i}^2] \le C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^3\theta_s\theta_t, & \text{if } (i',s',\ell')=(i,s,\ell),\ (k',t')=(k,t);\\ C\alpha^2\|\theta\|^4\theta_k\theta_t\theta_\ell^2\,\mathbb{E}[W_{is}^2W_{kt}^2W_{\ell i}^2] \le C\alpha^2\|\theta\|^4\theta_i^2\theta_k^2\theta_\ell^3\theta_s\theta_t^2, & \text{if } (i',s',\ell')=(i,s,\ell),\ (k',t')=(t,k);\\ C\alpha^2\|\theta\|^4\theta_k^2\theta_\ell\theta_s\,\mathbb{E}[W_{is}^2W_{kt}^2W_{\ell i}^2] \le C\alpha^2\|\theta\|^4\theta_i^2\theta_k^3\theta_\ell^2\theta_s^2\theta_t, & \text{if } (i',s',\ell')=(i,\ell,s),\ (k',t')=(k,t);\\ C\alpha^2\|\theta\|^4\theta_k\theta_t\theta_\ell\theta_s\,\mathbb{E}[W_{is}^2W_{kt}^2W_{\ell i}^2] \le C\alpha^2\|\theta\|^4\theta_i^2\theta_k^2\theta_\ell^2\theta_s^2\theta_t^2, & \text{if } (i',s',\ell')=(i,\ell,s),\ (k',t')=(t,k);\\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\mathrm{Var}\Big(\frac{1}{v}\sum_{\substack{i,k,\ell(dist)\\ s\notin\{i,\ell\},t\neq k,\\ (s,t)\neq(k,i)}}\beta_{ik\ell}W_{is}W_{kt}W_{\ell i}\Big) \le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\sum_{\substack{i,k,\ell,\\ s,t}}(\theta_i^2\theta_k^3\theta_\ell^3\theta_s\theta_t + \theta_i^2\theta_k^2\theta_\ell^3\theta_s\theta_t^2 + \theta_i^2\theta_k^2\theta_\ell^2\theta_s^2\theta_t^2)$$

$$\le \frac{C\alpha^2\|\theta\|^4}{\|\theta\|_1^4}\big(\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^2 + \|\theta\|^6\|\theta\|_3^3\|\theta\|_1 + \|\theta\|^{10}\big)$$

$$\le \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^6}{\|\theta\|_1^2}.$$

We plug the above results into Z_{3c}^*. Since $\|\theta\|^2 \le \|\theta\|_1\theta_{\max} \ll \|\theta\|_1^2$, we have $\frac{C\alpha^2\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^4} \ll \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^6}{\|\theta\|_1^2}$. It follows that

$$\mathrm{Var}(Z_{3c}^*) \le \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^6}{\|\theta\|_1^2}.$$

Since $\|\theta\|_3^6 \ll \|\theta\|_3^3\|\theta\|_1$, the variance of Z_{3c}^* is dominated by the variance of $\widetilde{Z}_{3c}$. It follows that

$$\mathrm{Var}(Z_{3c}) \le 2\mathrm{Var}(\widetilde{Z}_{3c}) + 2\mathrm{Var}(Z_{3c}^*) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1}. \tag{114}$$

Last, we study Z_{3d}. In the definition of Z_{3d}, if we switch the two indices (j,k), then it becomes

$$Z_{3d} = \sum_{\substack{i,j,k,\ell\\(dist)}}(\eta_i-\tilde\eta_i)\eta_k(\eta_k-\tilde\eta_k)\eta_j\widetilde\Omega_{j\ell}W_{\ell i} = \sum_{\substack{i,j,k,\ell\\(dist)}}(\eta_k\eta_j\widetilde\Omega_{j\ell})(\eta_i-\tilde\eta_i)(\eta_k-\tilde\eta_k).$$

At the same time, we recall that

$$Z_{3c} = \sum_{\substack{i,j,k,\ell\\(dist)}}(\eta_i-\tilde\eta_i)\eta_j^2(\eta_k-\tilde\eta_k)\widetilde\Omega_{k\ell}W_{\ell i} = \sum_{\substack{i,j,k,\ell\\(dist)}}(\eta_j^2\widetilde\Omega_{k\ell})(\eta_i-\tilde\eta_i)(\eta_k-\tilde\eta_k).$$

Here, Z_{3d} has a similar structure as Z_{3c}. Moreover, in deriving the bound for $\mathrm{Var}(Z_{3c})$, we have used $|\eta_j^2\widetilde\Omega_{k\ell}| \le C\alpha\theta_j^2\theta_k\theta_\ell$. In the expression of Z_{3d} above, we also have $|\eta_k\eta_j\widetilde\Omega_{j\ell}| \le C\alpha\theta_j^2\theta_k\theta_\ell$. Therefore, we can use (113) and (114) to directly get

$$\mathbb{E}[Z_{3d}] = 0, \qquad \mathrm{Var}(Z_{3d}) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1} \tag{115}$$

Now, we combine (107), (110), (113) and (114) to get

$$\mathbb{E}[Z_3] = 0.$$

We also combine (108), (111), (114)-(115). Since $\|\theta\|^4 \leq \|\theta\|_1 \|\theta\|_3^3$, the right hand side of (114)-(115) is dominated by the right hand side of (111); since $\|\theta\|_3^6 \ll \|\theta\|_1^2$, the right hand side of (108) is negligible to the right hand side of (111). It follows that

$$\mathrm{Var}(Z_3) \leq C\alpha^2 \|\theta\|^4 \|\theta\|_3^6.$$

This proves the claims of Z_3.

Next, we analyze Z_4. Since $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$,

$$Z_4 = \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i}$$
$$+ \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j\widetilde{\Omega}_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} + \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j\widetilde{\Omega}_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i}.$$

If we relabel (i, j, k, ℓ) as (ℓ', k', j', i') in the last sum, it is equal to the first sum. Therefore,

$$\begin{aligned} Z_4 = 2 &\sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} \\ &+ \sum_{i,j,k,\ell(dist)} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i} \\ &+ \sum_{i,j,k,\ell(dist)} (\eta_i - \tilde{\eta}_i)\eta_j\widetilde{\Omega}_{jk}\eta_k(\eta_\ell - \tilde{\eta}_\ell)W_{\ell i} \\ &\equiv Z_{4a} + Z_{4b} + Z_{4c}. \end{aligned} \tag{116}$$

First, we study Z_{4a} and Z_{4b}. We show that they have the same structure as Z_{3c} and Z_{3a}, respectively. In Z_{4a}, by relabeling (i, j, k, ℓ) as (ℓ, k, j, i), we have

$$Z_{4a} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_\ell(\eta_k - \tilde{\eta}_k)\widetilde{\Omega}_{kj}\eta_j(\eta_i - \tilde{\eta}_i)W_{i\ell} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_j\eta_\ell\widetilde{\Omega}_{kj})(\eta_i - \tilde{\eta}_i)(\eta_k - \tilde{\eta}_k)W_{\ell i}.$$

At the same time, we recall the definition of Z_{3c} in (106):

$$Z_{3c} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i - \tilde{\eta}_i)\eta_j^2(\eta_k - \tilde{\eta}_k)\widetilde{\Omega}_{k\ell}W_{\ell i} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_j^2\widetilde{\Omega}_{k\ell})(\eta_i - \tilde{\eta}_i)(\eta_k - \tilde{\eta}_k)W_{\ell i}.$$

It is seen that Z_{4a} has a similar structure as Z_{3c} does. Also, by (74) and (81), in the expression of Z_{4a}, we have $|\eta_j\eta_\ell\widetilde{\Omega}_{kj}| \leq C\alpha\theta_j^2\theta_k\theta_\ell$, while in the expression of Z_{3d}, we have $|\eta_j^2\widetilde{\Omega}_{k\ell}| \leq C\alpha\theta_j^2\theta_k\theta_\ell$. As a result, if we use similar calculation as before, we will get the same conclusion for Z_{4a} and Z_{3d}. Hence, we use (113)-(114) to conclude that

$$\mathbb{E}[Z_{4a}] = 0, \qquad \mathrm{Var}(Z_{4a}) \leq \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1}. \tag{117}$$

For Z_{4b}, we note that

$$Z_{4b} = \sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde{\eta}_j)\widetilde{\Omega}_{jk}(\eta_k - \tilde{\eta}_k)\eta_\ell W_{\ell i} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_\ell\widetilde{\Omega}_{jk})(\eta_j - \tilde{\eta}_j)(\eta_k - \tilde{\eta}_k)W_{\ell i},$$

where $|\eta_i\eta_\ell\widetilde{\Omega}_{jk}| \le C\alpha\theta_i\theta_j\theta_k\theta_\ell$. At the same time, we recall the definition of Z_{3a} in (106):

$$Z_{3a} = \sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde{\eta}_j)\eta_j(\eta_k - \tilde{\eta}_k)\widetilde{\Omega}_{k\ell}W_{\ell i} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_j\widetilde{\Omega}_{k\ell})(\eta_j - \tilde{\eta}_j)(\eta_k - \tilde{\eta}_k)W_{\ell i},$$

where $|\eta_i\eta_j\widetilde{\Omega}_{k\ell}| \le C\alpha\theta_i\theta_j\theta_k\theta_\ell$. Therefore, we can quote the results for Z_{3a} in (107)-(108) to get

$$\mathbb{E}[Z_{4b}] = 0, \qquad \mathrm{Var}(Z_{4b}) \le \frac{C\alpha^2\|\theta\|_3^{12}}{\|\theta\|_1^2}. \tag{118}$$

Second, we study Z_{4c}. It is seen that

$$\begin{aligned} Z_{4c} &= \sum_{i,j,k,\ell(dist)} \Big(-\frac{1}{\sqrt{v}}\sum_{s\neq i} W_{is}\Big)\eta_j\widetilde{\Omega}_{jk}\eta_k\Big(-\frac{1}{\sqrt{v}}\sum_{t\neq \ell} W_{\ell t}\Big)W_{\ell i} \\ &= \frac{1}{v}\sum_{\substack{i,\ell(dist)\\ s\neq i, t\neq \ell}} \Big(\sum_{j,k(dist)\notin\{i,\ell\}} \eta_j\eta_k\widetilde{\Omega}_{jk}\Big)W_{is}W_{\ell t}W_{\ell i} \\ &\equiv \frac{1}{v}\sum_{\substack{i,\ell(dist)\\ s\neq i, t\neq \ell}} \beta_{i\ell}W_{is}W_{\ell t}W_{\ell i}, \end{aligned}$$

where

$$|\beta_{i\ell}| \le \sum_{j,k(dist)\notin\{i,\ell\}} |\eta_j\eta_k\widetilde{\Omega}_{jk}| \le \sum_{j,k} C\alpha\theta_j^2\theta_k^2 \le C\alpha\|\theta\|^4. \tag{119}$$

We divide the summands into four groups: (i) $s=\ell$, $t=i$; (ii) $s=\ell$, $t\neq i$; (iii) $s\neq\ell$, $t=i$; (iv) $s\neq\ell$, $t\neq i$. By symmetry, the sum of group (ii) and the sum of group (iii) are equal. It yields that

$$\begin{aligned} Z_{4c} = &\frac{1}{v}\sum_{i,\ell(dist)} \beta_{i\ell}W_{\ell i}^3 + \frac{2}{v}\sum_{\substack{i,\ell(dist)\\ s\notin\{i,\ell\}}} \beta_{i\ell}W_{is}W_{\ell i}^2 + \frac{1}{v}\sum_{\substack{i,\ell(dist)\\ s\notin\{i,\ell\}, t\notin\{\ell,i\}}} \beta_{i\ell}W_{is}W_{\ell t}W_{\ell i} \\ &\equiv \widetilde{Z}_{4c} + Z_{4c}^* + Z_{4c}^\dagger. \end{aligned}$$

Only $\widetilde{Z}_{4c}$ has a nonzero mean. By (80) and (119),

$$\big|\mathbb{E}[Z_{4c}]\big| = \big|\mathbb{E}[\widetilde{Z}_{4c}]\big| \le \frac{C}{\|\theta\|_1^2}\sum_{i,\ell} \alpha\|\theta\|^4\theta_i\theta_\ell \le C\alpha\|\theta\|^4. \tag{120}$$

We now compute the variances of these terms. It is seen that

$$\mathrm{Var}(\widetilde{Z}_{4c}) \le \frac{C}{v^2}\sum_{i,\ell(dist)} \beta_{i\ell}^2\mathrm{Var}(W_{i\ell}^3) \le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^4}\sum_{i,\ell}\theta_i\theta_\ell \le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^2}.$$

For Z_{4c}^*, by direct calculations,

$$\begin{aligned} &\beta_{i\ell}\beta_{i'\ell'}\cdot\mathbb{E}[W_{is}W_{\ell i}^2\cdot W_{i's'}W_{\ell' i'}^2] \\ &\le C\alpha^2\|\theta\|^8\cdot\mathbb{E}[W_{is}W_{\ell i}^2\cdot W_{i's'}W_{\ell' i'}^2] \\ &\le \begin{cases} C\alpha^2\|\theta\|^8\cdot\mathbb{E}[W_{is}^2W_{\ell i}^4] \le C\alpha^2\|\theta\|^8\theta_i^2\theta_\ell\theta_s, & \text{if } i'=i,\ s'=s,\ \ell'=\ell; \\ C\alpha^2\|\theta\|^8\cdot\mathbb{E}[W_{is}^2W_{\ell i}^2W_{\ell' i}^2] \le C\alpha^2\|\theta\|^8\theta_i^3\theta_\ell\theta_s\theta_{\ell'}, & \text{if } i'=i,\ s'=s,\ \ell'\neq\ell; \\ C\alpha^2\|\theta\|^8\cdot\mathbb{E}[W_{is}^3W_{\ell i}^3] \le C\alpha^2\|\theta\|^8\theta_i^2\theta_\ell\theta_s, & \text{if } i'=i,\ s'=\ell,\ \ell'=s; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Var}(Z_{4c}^*) &\le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^4}\Big(\sum_{i,\ell,s}\theta_i^2\theta_\ell\theta_s + \sum_{i,\ell,s,\ell'}\theta_i^3\theta_\ell\theta_s\theta_{\ell'}\Big)\\ &\le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^4}\big(\|\theta\|^2\|\theta\|_1^2 + \|\theta\|_3^3\|\theta\|_1^3\big)\\ &\le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1},\end{aligned}$$

where, to get the last line, we have used $\|\theta\|^2 \ll \|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$. Regarding the variance of $Z_{4c}^\dagger$, we note that $W_{is}W_{\ell t}W_{\ell i}$ and $W_{i's'}W_{\ell' t'}W_{\ell' i'}$ are correlated only when the two undirected paths s-i-ℓ-t and s'-i'-ℓ'-t' are the same. Mimicking the argument in (85) or (90), we can derive that

$$\begin{aligned}\mathrm{Var}(Z_{4c}^\dagger) &\le \frac{C}{v^2}\sum_{\substack{i,\ell(dist)\\ s\notin\{i,\ell\},t\notin\{\ell,i\}}}\beta_{i\ell}^2\cdot\mathrm{Var}(W_{is}W_{\ell t}W_{\ell i})\\ &\le \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^4}\sum_{i,\ell,s,t}\theta_i^2\theta_\ell^2\theta_s\theta_t\\ &\le \frac{C\alpha^2\|\theta\|^{12}}{\|\theta\|_1^2}.\end{aligned}$$

Since $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$, the variance of $Z_{4c}^\dagger$ is dominated by the variance of Z_{4c}^*. Since $\|\theta\| \to \infty$, we have $\|\theta\|_3^3 \gg 1/\|\theta\|_1$; it follows that the variance of $\widetilde{Z}_{4c}$ is dominated by the variance of Z_{4c}^*. Combining the above gives

$$\mathrm{Var}(Z_{4c}) \le 3\mathrm{Var}(\widetilde{Z}_{4c}) + 3\mathrm{Var}(Z_{4c}^*) + 3\mathrm{Var}(Z_{4c}^\dagger) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1}. \tag{121}$$

We combine (117), (118) and (120) to get

$$\big|\mathbb{E}[Z_4]\big| \le C\alpha\|\theta\|^4 = o(\alpha^4\|\theta\|^8).$$

We then combine (117), (118) and (121). Since $\|\theta\|_3^6 \le (\theta_{\max}^2\|\theta\|_1)(\theta_{\max}\|\theta\|^2) = o(\|\theta\|_1\|\theta\|^2)$, the variance of Z_{4b} is negligible compared to the variances of Z_{4a} and Z_{4c}. It follows that

$$\mathrm{Var}(Z_4) \le \frac{C\alpha^2\|\theta\|^8\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8).$$

This proves the claims of Z_4.

Next, we analyze Z_5. By plugging in the definition of δ_{ij}, we have

$$\begin{aligned}Z_5 = &\sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde\eta_j)\eta_j(\eta_k-\tilde\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i} + \sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde\eta_j)^2\eta_k\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}\\ &+ \sum_{i,j,k,\ell(dist)}(\eta_i-\tilde\eta_i)\eta_j^2(\eta_k-\tilde\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i} + \sum_{i,j,k,\ell(dist)}(\eta_i-\tilde\eta_i)\eta_j(\eta_j-\tilde\eta_j)\eta_k\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}\\ = &\,2\sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde\eta_j)\eta_j(\eta_k-\tilde\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i} + \sum_{i,j,k,\ell(dist)}\eta_i(\eta_j-\tilde\eta_j)^2\eta_k\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}\\ &+ \sum_{i,j,k,\ell(dist)}(\eta_i-\tilde\eta_i)\eta_j^2(\eta_k-\tilde\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}\end{aligned}$$

(122)

$$\equiv Z_{5a} + Z_{5b} + Z_{5c}.$$

First, we study Z_{5a}. By definition, $(\tilde{\eta}_i - \eta_i)$ has the expression in (77). It follows that

$$\begin{aligned} Z_{5a} &= 2 \sum_{i,j,k,\ell (dist)} \eta_i \Big(-\frac{1}{\sqrt{v}} \sum_{s\neq j} W_{js}\Big) \eta_j \Big(-\frac{1}{\sqrt{v}} \sum_{t\neq k} W_{kt}\Big) \widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} \\ &= \frac{2}{v} \sum_{\substack{j,k(dist) \\ s\neq j, t\neq k}} \Big(\sum_{i,\ell (dist)\notin\{j,k\}} \eta_i\eta_j \widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}\Big) W_{js}W_{kt} \\ &\equiv \frac{2}{v} \sum_{\substack{j,k(dist) \\ s\neq j, t\neq k}} \beta_{jk} W_{js} W_{kt}, \end{aligned}$$

where

$$|\beta_{jk}| \leq \sum_{i,\ell(dist)\notin\{j,k\}} |\eta_i\eta_j\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}| \leq \sum_{i,\ell} (C\theta_i\theta_j)(C\alpha^2\theta_k\theta_\ell^2\theta_i) \leq C\alpha^2\|\theta\|^4\theta_j\theta_k. \tag{123}$$

In Z_{5a}, the summand has a nonzero mean only if $(s,t) = (k,j)$. We further decompose Z_{5a} into

$$Z_{5a} = \frac{2}{v} \sum_{j,k(dist)} \beta_{jk} W_{jk}^2 + \frac{2}{v} \sum_{\substack{j,k(dist) \\ s\neq j, t\neq k, \\ (s,t)\neq(k,j)}} \beta_{jk} W_{js}W_{kt} \equiv \widetilde{Z}_{5a} + Z_{5a}^*.$$

Only the first term has a nonzero mean. By (80) and (123), we have

$$\big|\mathbb{E}[Z_{5a}]\big| = \big|\mathbb{E}[\widetilde{Z}_{5a}]\big| \leq \frac{C}{\|\theta\|_1^2} \sum_{j,k} (\alpha^2\|\theta\|^4\theta_j\theta_k)(\theta_j\theta_k) \leq \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^2}. \tag{124}$$

We then compute the variances. In each of $\widetilde{Z}_{5a}$ and Z_{5a}^*, two summands are uncorrelated unless they are exactly the same variables (e.g., when $(j',k') = (k,j)$ in $\widetilde{Z}_{5a}$). Mimicking the argument in (85) or (90), we can derive that

$$\mathrm{Var}(\widetilde{Z}_{5a}) \leq \frac{C}{v^2} \sum_{j,k(dist)} \beta_{jk}^2 \,\mathrm{Var}(W_{jk}^2) \leq \frac{C\alpha^4\|\theta\|^8}{\|\theta\|_1^4} \sum_{j,k} (\theta_j^2\theta_k^2)\theta_j\theta_k \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^4},$$

$$\mathrm{Var}(Z_{5a}^*) \leq \frac{C}{v^2} \sum_{\substack{j,k(dist) \\ s\neq j, t\neq k, \\ (s,t)\neq(k,j)}} \beta_{jk}^2 \,\mathrm{Var}(W_{js}W_{kt}) \leq \frac{C\alpha^4\|\theta\|^8}{\|\theta\|_1^4} \sum_{j,k} (\theta_j^2\theta_k^2)\theta_j\theta_s\theta_k\theta_t \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^2}.$$

It immediately leads to

$$\mathrm{Var}(Z_{5a}) \leq 2\mathrm{Var}(\widetilde{Z}_{5a}) + 2\mathrm{Var}(Z_{5a}^*) \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^2}. \tag{125}$$

Second, we study Z_{5b}. It is seen that

$$Z_{5b} = \sum_{i,j,k,\ell(dist)} \eta_i \Big(-\frac{1}{\sqrt{v}} \sum_{s\neq j} W_{js}\Big) \Big(-\frac{1}{\sqrt{v}} \sum_{t\neq j} W_{jt}\Big) \eta_k \widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}$$

$$= \frac{1}{v} \sum_{j, s\neq j, t\neq j} \Big(\sum_{i,k,\ell (dist) \notin \{j\}} \eta_i \eta_k \widetilde{\Omega}_{k\ell} \widetilde{\Omega}_{\ell i} \Big) W_{js} W_{jt}$$

$$\equiv \frac{1}{v} \sum_{j, s\neq j, t\neq j} \beta_j W_{js} W_{jt},$$

where

(126) $$|\beta_j| \leq \sum_{i,k,\ell (dist)\notin\{j\}} |\eta_i\eta_k \widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}| \leq \sum_{i,k,\ell} (C\theta_i\theta_k)(C\alpha^2\theta_i\theta_k\theta_\ell^2) \leq C\alpha^2\|\theta\|^6.$$

In Z_{5b}, the summand has a nonzero mean only if $s = t$. We further decompose Z_{5b} into

$$Z_{5b} = \frac{1}{v}\sum_{j,s(dist)} \beta_j W_{js}^2 + \frac{1}{v} \sum_{\substack{j \\ s,t(dist)\notin\{j\}}} \beta_j W_{js}W_{jt} \equiv \widetilde{Z}_{5b} + Z_{5b}^*.$$

Only $\widetilde{Z}_{5b}$ has a nonzero mean. By (80) and (126),

(127) $$\big|\mathbb{E}[Z_{5b}]\big| = \big|\mathbb{E}[\widetilde{Z}_{5b}]\big| \leq \frac{C}{\|\theta\|_1^2}\sum_{j,s}(\alpha^2\|\theta\|^6)\theta_j\theta_s \leq C\alpha^2\|\theta\|^6.$$

To compute the variance, we note that in each of $\widetilde{Z}_{5b}$ and Z_{5b}^*, two summands are uncorrelated unless they are exactly the same random variables (e.g., when $\{j', s'\} = \{s, j\}$ in $\widetilde{Z}_{5b}$, and when $j' = j$ and $\{s', t'\} = \{s, t\}$ in Z_{5b}^*). Mimicking the argument in (85) or (90), we can derive that

$$\mathrm{Var}(\widetilde{Z}_{5b}) \leq \frac{C}{v^2}\sum_{j,s(dist)} \beta_j^2\,\mathrm{Var}(W_{js}^2) \leq \frac{C\alpha^4\|\theta\|^{12}}{\|\theta\|_1^4}\sum_{j,s}\theta_j\theta_s \leq \frac{C\alpha^4\|\theta\|^{12}}{\|\theta\|_1^2},$$

$$\mathrm{Var}(Z_{5b}^*) \leq \frac{C}{v^2}\sum_{\substack{j \\ s,t(dist)\notin\{j\}}} \beta_j^2\,\mathrm{Var}(W_{js}W_{jt}) \leq \frac{C\alpha^4\|\theta\|^{12}}{\|\theta\|_1^4}\sum_{j,s,t}\theta_j^2\theta_s\theta_t \leq \frac{C\alpha^4\|\theta\|^{14}}{\|\theta\|_1^2}.$$

Combining the above gives

(128) $$\mathrm{Var}(Z_{5b}) \leq 2\mathrm{Var}(\widetilde{Z}_{5b}) + 2\mathrm{Var}(Z_{5b}^*) \leq \frac{C\alpha^4\|\theta\|^{14}}{\|\theta\|_1^2}.$$

Third, we study Z_{5c}. If we relabel $(i, j, k, \ell) = (j, i, k, \ell)$, then Z_{5c} becomes

$$Z_{5c} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_j - \tilde{\eta}_j)\eta_i^2(\eta_k - \tilde{\eta}_k)\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell j} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i^2\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell j})(\eta_j - \tilde{\eta}_j)(\eta_k - \tilde{\eta}_k),$$

where $|\eta_i^2\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell j}| \leq C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2$. At the same time, we recall that

$$Z_{5a} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde{\eta}_j)\eta_j(\eta_k - \tilde{\eta}_k)\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_j\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i})(\eta_j - \tilde{\eta}_j)(\eta_k - \tilde{\eta}_k),$$

where $|\eta_i\eta_j\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}| \leq C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2$. It is easy to see that Z_{5c} has a similar structure as Z_{5c}. As a result, from (124)-(125), we immediately have

(129) $$\big|\mathbb{E}[Z_{5c}]\big| \leq \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^2}, \qquad \mathrm{Var}(Z_{5c}) \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^2}.$$

We now combine the results for Z_{5a}-Z_{5c}. Since $\|\theta\|^2 \leq \theta_{\max}\|\theta\|_1 \ll \|\theta\|_1^2$, $\mathbb{E}[Z_{5a}]$ and $\mathbb{E}[Z_{5c}]$ are of a smaller order than the the right hand side of (127). Since $\|\theta\|_3^6 \leq \theta_{\max}^2\|\theta\|^4 \ll \|\theta\|^6$, $\mathrm{Var}(Z_{5a})$ and $\mathrm{Var}(Z_{5c})$ are of a smaller order than the right hand side of (128). It follows that

$$\big|\mathbb{E}[Z_5]\big| \leq C\alpha^2\|\theta\|^6 = o(\alpha^4\|\theta\|^8), \qquad \mathrm{Var}(Z_5) \leq \frac{C\alpha^4\|\theta\|^{14}}{\|\theta\|_1^2} = o(\alpha^6\|\theta\|^8\|\theta\|_3^6).$$

We briefly explain why $\mathrm{Var}(Z_5) = o(\alpha^6\|\theta\|^8\|\theta\|_3^6)$: since $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, we immediately have $\|\theta\|^{14} \leq \|\theta\|^6(\|\theta\|_1\|\theta\|_3^3)^2$; it follows that the bound for $\mathrm{Var}(Z_5)$ is $\leq C\alpha^4\|\theta\|^6\|\theta\|_3^6$; note that $\alpha\|\theta\| \to \infty$, we immediately have $\alpha^4\|\theta\|^6\|\theta\|_3^6 = o(\alpha^6\|\theta\|^8\|\theta\|_3^6)$. This proves the claims of Z_5.

Last, we analyze Z_6. Plugging in the definition of δ_{ij}, we have

$$\begin{aligned} Z_6 = &\sum_{i,j,k,\ell (dist)} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{jk}\eta_k(\eta_\ell - \tilde\eta_\ell)\widetilde\Omega_{\ell i} + \sum_{i,j,k,\ell (dist)} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{jk}(\eta_k - \tilde\eta_k)\eta_\ell\widetilde\Omega_{\ell i} \\ &+ \sum_{i,j,k,\ell (dist)} (\eta_i - \tilde\eta_i)\eta_j\widetilde\Omega_{jk}\eta_k(\eta_\ell - \tilde\eta_\ell)\widetilde\Omega_{\ell i} + \sum_{i,j,k,\ell (dist)} (\eta_i - \tilde\eta_i)\eta_j\widetilde\Omega_{jk}(\eta_k - \tilde\eta_k)\eta_\ell\widetilde\Omega_{\ell i} \\ = &\, 2\sum_{i,j,k,\ell (dist)} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{jk}\eta_k(\eta_\ell - \tilde\eta_\ell)\widetilde\Omega_{\ell i} + 2\sum_{i,j,k,\ell (dist)} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{jk}(\eta_k - \tilde\eta_k)\eta_\ell\widetilde\Omega_{\ell i} \\ \equiv &\, Z_{6a} + Z_{6b}. \end{aligned}$$

By relabeling (i,j,k,ℓ) as (i,j,ℓ,k), we can write

$$Z_{6a} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{j\ell}\eta_\ell(\eta_k - \tilde\eta_k)\widetilde\Omega_{ki} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_\ell\widetilde\Omega_{j\ell}\widetilde\Omega_{ki})(\eta_j - \tilde\eta_j)(\eta_k - \tilde\eta_k),$$

where $|\eta_i\eta_\ell\widetilde\Omega_{j\ell}\widetilde\Omega_{ki}| \leq C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2$. Also, we write

$$Z_{6b} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde\eta_j)\widetilde\Omega_{jk}(\eta_k - \tilde\eta_k)\eta_\ell\widetilde\Omega_{\ell i} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_\ell\widetilde\Omega_{jk}\widetilde\Omega_{\ell i})(\eta_j - \tilde\eta_j)(\eta_k - \tilde\eta_k).$$

where $|\eta_i\eta_\ell\widetilde\Omega_{jk}\widetilde\Omega_{\ell i}| \leq C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2$. At the same time, we recall that

$$Z_{5a} = 2\sum_{\substack{i,j,k,\ell \\ (dist)}} \eta_i(\eta_j - \tilde\eta_j)\eta_j(\eta_k - \tilde\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i} = \sum_{\substack{i,j,k,\ell \\ (dist)}} (\eta_i\eta_j\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i})(\eta_j - \tilde\eta_j)(\eta_k - \tilde\eta_k),$$

where $|\eta_i\eta_j\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}| \leq C\alpha^2\theta_i^2\theta_j\theta_k\theta_\ell^2$. It is clear that both Z_{6a} and Z_{6b} have a similar structure as Z_{5a}. From (124)-(125), we immediately have

$$\big|\mathbb{E}[Z_6]\big| \leq \frac{C\alpha^2\|\theta\|^8}{\|\theta\|_1^2} = o(\alpha^4\|\theta\|^8), \qquad \mathrm{Var}(Z_6) \leq \frac{C\alpha^4\|\theta\|^8\|\theta\|_3^6}{\|\theta\|_1^2} = o(\|\theta\|^8).$$

This proves the claims of Z_6.

G.4.8. *Proofs of Lemmas G.8 and G.9.* Recall that $\lambda_1, \lambda_2, \ldots, \lambda_K$ are all the nonzero eigenvalues of Ω, arranged in the descending order in magnitude. Write for short $\alpha = |\lambda_2|/|\lambda_1|$. We shall repeatedly use the following results, which are proved in (74), (80), and (81):

$$v \asymp \|\theta\|_1^2, \qquad 0 < \eta_i < C\theta_i, \qquad |\widetilde\Omega_{ij}| \leq C\alpha\theta_i\theta_j.$$

Recall that $U_c = 4T_1 + F$, under the null hypothesis; $U_c = 4T_1 + 4T_2 + F$ under the alternative hypothesis. By definition,

$$T_1 = \sum_{i_1,i_2,i_3,i_4(dist)} \delta_{i_1i_2}\delta_{i_2i_3}\delta_{i_3i_4}W_{i_4i_1},$$

$$T_2 = \sum_{i_1,i_2,i_3,i_4(dist)} \delta_{i_1i_2}\delta_{i_2i_3}\delta_{i_3i_4}\widetilde{\Omega}_{i_4i_1},$$

$$F = \sum_{i_1,i_2,i_3,i_4(dist)} \delta_{i_1i_2}\delta_{i_2i_3}\delta_{i_3i_4}\delta_{i_4i_1},$$

where $\delta_{ij} = \eta_i(\eta_j - \widetilde{\eta}_j) + \eta_j(\eta_i - \widetilde{\eta}_i)$, for $1 \le i, j \le n$, $i \neq j$. By symmetry and elementary algebra, we further write

$$T_1 = 2T_{1a} + 2T_{1b} + 2T_{1c} + 2T_{1d}, \tag{130}$$

where

$$T_{1a} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_3} - \tilde{\eta}_{i_3})\big] \cdot W_{i_4i_1},$$

$$T_{1b} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_4} - \tilde{\eta}_{i_4})\big] \cdot W_{i_4i_1},$$

$$T_{1c} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_2} - \tilde{\eta}_{i_2})^2(\eta_{i_3} - \tilde{\eta}_{i_3})\big] \cdot W_{i_4i_1},$$

$$T_{1d} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}^2\big[(\eta_{i_2} - \tilde{\eta}_{i_2})^2(\eta_{i_4} - \tilde{\eta}_{i_4})\big] \cdot W_{i_4i_1}.$$

Similarly, we write

$$T_2 = 2T_{2a} + 2T_{2b} + 2T_{2c} + 2T_{2d}, \tag{131}$$

where

$$T_{2a} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_3} - \tilde{\eta}_{i_3})\big] \cdot \widetilde{\Omega}_{i_4i_1},$$

$$T_{2b} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_4} - \tilde{\eta}_{i_4})\big] \cdot \widetilde{\Omega}_{i_4i_1},$$

$$T_{2c} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_2} - \tilde{\eta}_{i_2})^2(\eta_{i_3} - \tilde{\eta}_{i_3})\big] \cdot \widetilde{\Omega}_{i_4i_1},$$

$$T_{2d} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}^2\big[(\eta_{i_2} - \tilde{\eta}_{i_2})^2(\eta_{i_4} - \tilde{\eta}_{i_4})\big] \cdot \widetilde{\Omega}_{i_4i_1}.$$

Also, similarly, we have

$$F = 2F_a + 12F_b + 2F_c, \tag{132}$$

where

$$F_a = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_3} - \tilde{\eta}_{i_3})(\eta_{i_4} - \tilde{\eta}_{i_4})\big],$$

$$F_b = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\eta_{i_4}\big[(\eta_{i_1}-\tilde{\eta}_{i_1})^2(\eta_{i_2}-\tilde{\eta}_{i_2})(\eta_{i_4}-\tilde{\eta}_{i_4})\big],$$

$$F_c = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}^2\eta_{i_4}^2\big[(\eta_{i_1}-\tilde{\eta}_{i_1})^2(\eta_{i_3}-\tilde{\eta}_{i_3})^2\big].$$

To show the lemmas, it is sufficient to show the following 11 items (a)-(k), corresponding to $T_{1a}, T_{1b}, T_{1c}, T_{1d}, T_{2a}, T_{2b}, T_{2c}, T_{2d}, F_a, F_b, F_c$, respectively. Item (a) claims that both under the null and the alternative,

$$|\mathbb{E}[T_{1a}]| \leq C\|\theta\|^6/\|\theta\|_1^2, \qquad \mathrm{Var}(T_{1a}) \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2. \tag{133}$$

Item (b) claims that both under the null and the alternative,

$$|\mathbb{E}[T_{1b}]| \leq C\|\theta\|^6/\|\theta\|_1^2,, \qquad \mathrm{Var}(T_{1b}) \leq C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1. \tag{134}$$

Item (c) claims that both under the null and the alternative,

$$\mathbb{E}[T_{1c}] = 0, \qquad \mathrm{Var}(T_{1c}) \leq C\|\theta\|_3^9/\|\theta\|_1, \tag{135}$$

Item (d) claims that

$$\mathbb{E}[T_{1d}] \asymp -\|\theta\|^4 \text{ under the null,}$$
$$|\mathbb{E}[T_{1d}]| \leq C\|\theta\|^4 \text{ under the alternative,} \tag{136}$$

and that both under the null and the alternative,

$$\mathrm{Var}(T_{1d}) \leq C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1. \tag{137}$$

Next, for item (e)-(h), we recall that under the null, $T_2 = 0$, and correspondingly $T_{2a} = T_{2b} = T_{2c} = T_{2d} = 0$, so we only need to consider the alternative. Recall that $\alpha = |\lambda_2/\lambda_1|$. Item (e) claims that under the alternative,

$$\mathbb{E}[T_{2a}] = 0, \qquad \mathrm{Var}(T_{2a}) \leq C\alpha^2 \cdot \|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^3. \tag{138}$$

Item (f) claims that under the alternative,

$$\mathbb{E}[T_{2b}] = 0, \qquad \mathrm{Var}(T_{2b}) \leq C\alpha^2 \cdot \|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5, \tag{139}$$

Item (g) claims that under the alternative,

$$|\mathbb{E}[T_{2c}]| \leq C\alpha\|\theta\|^6/\|\theta\|_1^3, \qquad \mathrm{Var}(T_{2c}) \leq C\alpha^2 \cdot \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1. \tag{140}$$

Item (h) claims that both under the null and the alternative,

$$|\mathbb{E}[T_{2d}]| \leq C\alpha\|\theta\|^6/\|\theta\|_1^3, \qquad \mathrm{Var}(T_{2d}) \leq C\alpha^2 \cdot \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1. \tag{141}$$

Finally, for items (i)-(k). Item (i) claims that both under the null and the alternative,

$$|\mathbb{E}[F_a]| \leq C\|\theta\|^8/\|\theta\|_1^4, \qquad \mathrm{Var}(F_a) \leq C\|\theta\|_3^{12}/\|\theta\|_1^4. \tag{142}$$

Item (j) claims that both under the null and the alternative,

$$|\mathbb{E}[F_b]| \leq C\|\theta\|^6/\|\theta\|_1^2, \qquad \mathrm{Var}(F_b) \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2. \tag{143}$$

Item (k) claims that

$$\mathbb{E}[F_c] \asymp \|\theta\|^4 \text{ under the null,}$$
$$|\mathbb{E}[F_c]| \leq C\|\theta\|^4 \text{ under the alternative,} \tag{144}$$

and that under both under the null and the alternative,

$$\mathrm{Var}(F_3) \le C\|\theta\|^{10}/\|\theta\|_1^2. \tag{145}$$

We now show Lemmas G.4 and G.5 follow once (a)-(k) are proved. In detail, first, we note that $\|\theta\|^6/\|\theta\|_1^2 = o(\|\theta\|^4)$. Inserting (136) and the first equation in each of (133)-(135) into (130) gives that

$$\mathbb{E}[T_1] \asymp -2\|\theta\|^4 \text{ under the null}, \qquad |\mathbb{E}[T_1]| \le C\|\theta\|^4 \text{ under the alternative},$$

and inserting (137) and the second equation in each of (133)-(135) into (130) gives that both under the null and the alternative,

$$\mathrm{Var}(T_1) \le C[\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2 + \|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|_3^9/\|\theta\|_1 + \|\theta\|^6\|\theta\|_3^3/\|\theta\|_1],$$

where since $\|\theta\|_3^3/\|\theta\|^2 = o(1)$ and $\|\theta\|^2/\|\theta\|_1 = o(1)$, the right hand side

$$\le C[\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1^2 + \|\theta\|^6\|\theta\|_3^3/\|\theta\|_1] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1.$$

Second, inserting the first equation in each of (138)-(141) into (131) gives that under the alternative (recall that $T_2 = 0$ under the null),

$$|\mathbb{E}[T_2]| \le C\alpha\|\theta\|^6/\|\theta\|_1^3,$$

and inserting the second equation in each of (138)-(141) into (131) gives

$$\mathrm{Var}(T_2) \le C\alpha^2[\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5] \le C\alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1,$$

where we have used $\|\theta\|^2 = o(\|\theta\|_1^2)$. Third, note that $\|\theta\|^8/\|\theta\|_1^4 = o(\|\theta\|^4)$ and $\|\theta\|^6/\|\theta\|_1^2 = o(\|\theta\|^4)$. Inserting (144) and the first equation in each of (142)-(143) into (132) gives

$$\mathbb{E}[F] \sim 2\|\theta\|^4 \text{ under the null}, \qquad |\mathbb{E}[F]| \le C\|\theta\|^4 \text{ under the alternative},$$

and inserting (145) and the second equation in each of (142)-(143) into (132) gives that both under the null and the alternative,

$$\mathrm{Var}(F) \le C[\|\theta\|_3^{12}/\|\theta\|_1^4 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2 + \|\theta\|^{10}/\|\theta\|_1^2] \le C\|\theta\|^{10}/\|\theta\|_1^2,$$

where we have used $\|\theta\|_3^3 \ll \theta\|^2 \ll \|\theta\|_1$ and $\|\theta\|_3^3/\|\theta\|^2 = o(1)$.

We now combine the above results for T_1, T_2 and F. First, since that $U_c = 4T_1 + F$ under the null, it follows that under the null,

$$\mathbb{E}[U_c] \sim -6\|\theta\|^4,$$

and

$$\mathrm{Var}(U_c) \le C[\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^{10}/\|\theta\|_1^2] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1,$$

where we have used $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$ (a direct use of Cauchy-Schwartz inequality). Second, since $U_c = 4T_1 + 4T_2 + F$ under the alternative, it follows that under the alternative,

$$|\mathbb{E}[U_c]| \le C\|\theta\|^4,$$

and

$$\mathrm{Var}(U_c) \le C[\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^{10}/\|\theta\|_1^2] \le C\|\theta\|^6\|\theta\|_3^3(\alpha^2\|\theta\|^2 + 1)/\|\theta\|_1,$$

where we have used $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$ and basic algebra. Combining the above gives all the claims in Lemmas G.4 and G.5.

It remains to show the 11 items (a)-(k). We consider them separately.

Consider Item (a). The goal is to show (133). Recall that

$$T_{1a} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1} - \tilde{\eta}_{i_1})(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_3} - \tilde{\eta}_{i_3})\big] \cdot W_{i_4 i_1},$$

and that

$$\widetilde{\eta} - \eta = v^{-1/2} W 1_n. \tag{146}$$

Plugging (146) into T_{11} gives

$$\begin{aligned} T_{1a} &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\Big(\sum_{j_1, j_1 \neq i_1} W_{i_1 j_1}\Big)\Big(\sum_{j_2, j_2 \neq i_2} W_{i_2 j_2}\Big)\Big(\sum_{j_3, j_3 \neq i_3} W_{i_3 j_3}\Big) W_{i_4 i_1} \\ &= -\frac{1}{v^{3/2}} \sum_{\substack{i_1,i_2,i_3,i_4(dist) \\ j_1 \neq i_1, j_2 \neq i_2, j_3 \neq i_3}} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 j_1} W_{i_2 j_2} W_{i_3 j_3} W_{i_1 i_4}. \end{aligned}$$

By basic combinatorics and careful observations, we have

$$W_{i_1 j_1} W_{i_2 j_2} W_{i_3 j_3} W_{i_1 i_4} = \begin{cases} W_{i_1 i_4}^2 W_{i_2 i_3}^2, & \text{if } j_1 = i_4, (j_2, j_3) = (i_3, i_2), \\ W_{i_1 i_4}^2 W_{i_2 j_2} W_{i_3 j_3}, & \text{if } j_1 = i_4, (j_2, j_3) \neq (i_3, i_2), \\ W_{i_2 i_3}^2 W_{i_1 j_1} W_{i_1 i_4}, & \text{if } j_1 \neq i_4, (j_2, j_3) = (i_3, i_2), \\ W_{i_1 i_2}^2 W_{i_3 j_3} W_{i_1 i_4}, & \text{if } (j_1, j_2) = (i_2, i_1), \\ W_{i_1 i_3}^2 W_{i_2 j_2} W_{i_1 i_4}, & \text{if } (j_1, j_3) = (i_3, i_1), \\ W_{i_1 j_1} W_{i_2 j_2} W_{i_3 j_3} W_{i_1 i_4}, & \text{otherwise.} \end{cases} \tag{147}$$

This allows us to further split T_{11} into 6 different terms:

$$T_{1a} = X_a + X_{b1} + X_{b2} + X_{b3} + X_{b4} + X_c, \tag{148}$$

where

$$\begin{aligned} X_a &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 i_4}^2 W_{i_2 i_3}^2, \\ X_{b1} &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2, j_3 \\ (j_2, j_3) \neq \{i_3, i_2\}}} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 i_4}^2 W_{i_2 j_2} W_{i_3 j_3}, \\ X_{b2} &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_1 (j_1 \neq i_4)} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_2 i_3}^2 W_{i_1 j_1} W_{i_1 i_4}, \\ X_{b3} &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_3 (j_3 \neq i_3)} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 i_2}^2 W_{i_3 j_3} W_{i_1 i_4}, \\ X_{b4} &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2 (j_2 \neq i_2)} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 i_3}^2 W_{i_2 j_2} W_{i_1 i_4}, \\ X_c &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1, j_2, j_3 \\ j_1 \notin \{i_1, i_4\}, (j_2, j_3) \neq (i_3, i_2) \\ (j_1, j_2) \neq (i_2, i_1), (j_1, j_3) \neq (i_3, i_1)}} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1 j_1} W_{i_2 j_2} W_{i_3 j_3} W_{i_1 i_4}. \end{aligned}$$

We now show (133). Consider the first claim of (133). It is seen that out of the 6 terms on the right hand side of (148), the mean of all terms are 0, except for the first term. Note that for any $1 \le i, j \le n$, $i \neq j$, $\mathbb{E}[W_{ij}^2] = \Omega_{ij}(1-\Omega_{ij})$, where Ω_{ij} are upper bounded by $o(1)$ uniformly for all such i, j. It follows

$$\mathbb{E}[X_a] = -v^{-3/2} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\mathbb{E}[W_{i_1i_4}^2]\mathbb{E}[W_{i_2i_3}^2]$$
$$= -(1+o(1)) \cdot v^{-3/2} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\Omega_{i_1i_4}\Omega_{i_2i_3}.$$

Since for any $1 \le i, j \le n$, $i \neq j$, $0 < \eta_i \le C\theta_i$, $\Omega_{ij} \le C\theta_i\theta_j$ and $v \asymp \|\theta\|_1^2$,

$$|\mathbb{E}[X_a]| \le C(\|\theta\|_1)^{-3} \sum_{i_1,i_2,i_3,i_4(dist)} \theta_{i_1}\theta_{i_2}^2\theta_{i_3}^2\theta_{i_4}^2 \le C\|\theta\|^6/\|\theta\|_1^2.$$

Inserting these into (148) gives

$$|\mathbb{E}[T_{1a}]| \le C\|\theta\|^6/\|\theta\|_1^2, \tag{149}$$

and the first claim of (133) follows.

Consider the second claim of (133) next. By (148) and Cauchy-Schwartz inequality,

$$\mathrm{Var}(T_{1a}) \le C\mathrm{Var}(X_a) + \mathrm{Var}(X_{b1}) + \mathrm{Var}(X_{b2}) + \mathrm{Var}(X_{b3}) + \mathrm{Var}(X_{b4}) + \mathrm{Var}(X_c))$$
$$\le C(\mathrm{Var}(X_a) + \mathbb{E}[X_{b1}^2] + \mathbb{E}[X_{b2}^2] + \mathbb{E}[X_{b3}^2] + \mathbb{E}[X_{b4}^2] + \mathbb{E}[X_c^2]). \tag{150}$$

We now consider $\mathrm{Var}(X_a)$, $\mathbb{E}[X_{b1}^2] + \mathbb{E}[X_{b2}^2] + \mathbb{E}[X_{b3}^2] + \mathbb{E}[X_{b4}^2]$, and $\mathbb{E}[X_c^2]$, separately.

Consider $\mathrm{Var}(X_a)$. Write $\mathrm{Var}(X_a)$ as

$$v^{-3} \sum_{\substack{i_1,\cdots,i_4(dist)\\ i_1',\cdots,i_4'(dist)}} \eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i_2'}\eta_{i_3'}\eta_{i_4'}$$
$$\mathbb{E}\big[(W_{i_1i_4}^2 W_{i_2i_3}^2 - \mathbb{E}[W_{i_1i_4}^2 W_{i_2i_3}^2])(W_{i_1'i_4'}^2 W_{i_2'i_3'}^2 - \mathbb{E}[W_{i_1'i_4'}^2 W_{i_2'i_3'}^2])\big]. \tag{151}$$

In the sum, a term is nonzero only when one of the following cases happens.

- (A). $\{W_{i_1i_4}, W_{i_2i_3}, W_{i_1'i_4'}, W_{i_2'i_3'}\}$ has 2 distinct random variables.
- (B). $\{W_{i_1i_4}, W_{i_2,i_3}, W_{i_1'i_4'}, W_{i_2'i_3'}\}$ has 3 distinct random variables. This has 4 sub-cases: (B1) $W_{i_1i_4} = W_{i_1'i_4'}$, (B2) $W_{i_1i_4} = W_{i_2'i_3'}$, (B3) $W_{i_2i_3} = W_{i_1'i_4'}$, and (B4) $W_{i_2i_3} = W_{i_2'i_3'}$.

For Case (A), the two sets $\{i_1, i_2, i_3, i_4\}$ and $\{i_1', i_2', i_3', i_4'\}$ are identical. By basic statistics and independence between $W_{i_1i_4}$ and $W_{i_2i_3}$,

$$\mathbb{E}[(W_{i_1i_4}^2 W_{i_2i_3}^2 - \mathbb{E}[W_{i_1i_4}^2 W_{i_2i_3}^2])(W_{i_1'i_4'}^2 W_{i_2'i_3'}^2 - \mathbb{E}[W_{i_1'i_4'}^2 W_{i_2'i_3'}^2])]$$
$$= \mathbb{E}[(W_{i_1i_4}^2 W_{i_2i_3}^2 - \mathbb{E}[W_{i_1i_4}^2 W_{i_2i_3}^2])^2]$$
$$= \mathbb{E}[W_{i_1i_4}^4]\mathbb{E}[W_{i_2i_3}^4] - (\mathbb{E}[W_{i_1i_4}^2])^2(\mathbb{E}[W_{i_2i_3}^2])^2$$
$$\le \mathbb{E}[W_{i_1i_4}^4]\mathbb{E}[W_{i_2i_3}^4], \tag{152}$$

where by basic statistics and that $\Omega_{ij} \le C\theta_i\theta_j$ for all $1 \le i, j \le n$, $i \le j$, the right hand side

$$\le C\Omega_{i_1i_4}\Omega_{i_2i_3} \le C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}.$$

Combining these with (151) and noting that $v \sim \|\theta\|_1^2$ and that $0 < \eta_i \le C\theta_i$ for all $1 \le i \le n$, the contribution of this case to $\mathrm{Var}(X_a)$ is no more than

$$C(\|\theta\|_1)^{-6} \sum_{i_1,\cdots,i_4(dist)} \sum_a \theta_{i_1}^{a_1+1}\theta_{i_2}^{a_2+2}\theta_{i_3}^{a_3+2}\theta_{i_4}^{a_4+2}, \tag{153}$$

where $a = (a_1, a_2, a_3, a_4)$ and each a_i is either 0 and 1, satisfying $a_1 + a_2 + a_3 + a_4 = 3$. Note that the right hand side of (153) is no greater than

$$C(\|\theta\|_1)^{-6} \max\{\|\theta\|_1 \|\theta\|_3^9, \|\theta\|^4 \|\theta\|_3^6\} \leq C\|\theta\|_3^9/\|\theta\|_1^5,$$

where we have used $\|\theta\|^4 \leq \|\theta\|_1 \|\theta\|_3^3$.

Next, consider (B1). By independence between $W_{i_1 i_4}$, $W_{i_2 i_3}$, and $W_{i_2' i_3'}$ and basic algebra,

$$\begin{aligned}
&\mathbb{E}[(W_{i_1 i_4}^2 W_{i_2 i_3}^2 - \mathbb{E}[W_{i_1 i_4}^2 W_{i_2 i_3}^2])(W_{i_1' i_4'}^2 W_{i_2' i_3'}^2 - \mathbb{E}[W_{i_1' i_4'}^2 W_{i_2' i_3'}^2])] \\
=&\mathbb{E}[(W_{i_1 i_4}^2 W_{i_2 i_3}^2 - \mathbb{E}[W_{i_1 i_4}^2 W_{i_2 i_3}^2])(W_{i_1 i_4}^2 W_{i_2' i_3'}^2 - \mathbb{E}[W_{i_1 i_4}^2 W_{i_2' i_3'}^2])] \\
=&\mathbb{E}[W_{i_1 i_4}^4]\mathbb{E}[W_{i_2 i_3}^2]\mathbb{E}[W_{i_2' i_3'}^2] - (\mathbb{E}[W_{i_1 i_4}^2])^2 \mathbb{E}[W_{i_2 i_3}^2]\mathbb{E}[W_{i_2' i_3'}^2] \\
=&\mathrm{Var}(W_{i_1 i_4}^2)\mathbb{E}[W_{i_2 i_3}^2]\mathbb{E}[W_{i_2' i_3'}^2], \qquad (154)
\end{aligned}$$

where by similar arguments, the last term

$$\leq C\Omega_{i_1 i_4}\Omega_{i_2 i_3}\Omega_{i_2' i_3'} \leq C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\theta_{i_2'}\theta_{i_3'}.$$

Combining this with (151) and using similar arguments, the contribution of this case to $\mathrm{Var}(X_a)$

$$\leq C(\|\theta\|_1)^{-6} \sum_{\substack{i_1, i_2, i_3, i_4 (dist) \\ i_2', i_3' (dist)}} C\theta_{i_1}^{b_1+1}\theta_{i_2}^2\theta_{i_3}^2\theta_{i_4}^{b_2+2}\theta_{i_2'}^2\theta_{i_3'}^2, \qquad (155)$$

where similarly b_1, b_2 are either 0 or 1 and $b_1 + b_2 = 1$. By similar argument, the right hand side

$$\leq C\|\theta\|_1 \|\theta\|^8 \|\theta\|_3^3/\|\theta\|_1^6 = C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5.$$

The discussion for (B2), (B3), and (B4) are similar so is omitted, and their contribution to $\mathrm{Var}(X_a)$ are respectively

$$\leq C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5, \qquad (156)$$

$$\leq C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5, \qquad (157)$$

and

$$\leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4. \qquad (158)$$

Finally, inserting (153), (155), (156), (157), and (158) into (151) gives

$$\mathrm{Var}(X_a) \leq C[\|\theta\|_3^9/\|\theta\|_1^5 + \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4] \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \qquad (159)$$

where we have used $\|\theta\|_3^3 \ll \|\theta\|^2$ and $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$.

Consider $\mathbb{E}[X_{b1}^2] + \mathbb{E}[X_{b21}^2] + \mathbb{E}[X_{b3}^2] + \mathbb{E}[X_{b4}^2]$. We claim that both under the null and the alternative,

$$\mathbb{E}[X_{b1}^2] \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2, \qquad (160)$$

$$\mathbb{E}[X_{b2}^2] \leq C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3, \qquad (161)$$

$$\mathbb{E}[X_{b3}^2] \leq C\|\theta\|^6\|\theta\|_3^6/\|\theta\|_1^4, \qquad (162)$$

$$\mathbb{E}[X_{b4}^2] \leq C\|\theta\|^6\|\theta\|_3^6/\|\theta\|_1^4, \qquad (163)$$

where the last two terms are seen to be negligible compared to the other two. Therefore,

$$\mathbb{E}[X_{b1}^2] + \mathbb{E}[X_{b2}^2] + \mathbb{E}[X_{b3}^2] + \mathbb{E}[X_{b4}^2] \leq C[\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2 + \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3], \qquad (164)$$

where since $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$ (Cauchy-Schwartz inequality) the right hand side

$$\leq C[\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2.$$

We now prove (160)-(163). Since the study for $\mathbb{E}[X_{b1}^2], \mathbb{E}[X_{b2}^2], \mathbb{E}[X_{b3}^2]$ and $\mathbb{E}[X_{b4}^2]$ are similar, we only present the proof for $\mathbb{E}[X_{b1}^2]$. Write $\mathbb{E}[X_{b1}^2]$ as

$$v^{-3} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i_1',i_2',i_3',i_4'(dist)}} \sum_{\substack{j_2,j_3\\ (j_2,j_3)\neq(i_3,i_2)}} \sum_{\substack{j_2',j_3'\\ (j_2',j_3')\neq(i_3',i_2')}} \eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i_2'}\eta_{i_3'}\eta_{i_4'}W_{i_1i_4}^2 W_{i_2j_2}W_{i_3j_3}W_{i_1'i_4'}^2 W_{i_2'j_2'}W_{i_3'j_3'}.$$

Consider the term

$$W_{i_1i_4}^2 W_{i_2j_2}W_{i_3j_3}W_{i_1'i_4'}^2 W_{i_2'j_2'}W_{i_3'j_3'}.$$

In order for the mean to be nonzero, we have two cases

- Case A. The two sets of random variables $\{W_{i_1i_4}, W_{i_2j_2}, W_{i_3j_3}\}$ and $\{W_{i_1'i_4'}, W_{i_2'j_2'}, W_{i_3'j_3'}\}$ are identical.
- Case B. The two sets $\{W_{i_2j_2}, W_{i_3j_3}\}$ and $\{W_{i_2'j_2'}, W_{i_3'j_3'}\}$ are identical.

Consider Case A. In this case, $\{i_2', i_3', i_4'\}$ are three distinct indices in $\{i_1, i_2, i_3, i_4, j_2, j_3\}$, and for some integers satisfying $0 \leq a_1, a_2, \ldots, a_6 \leq 1$, $a_1 + a_2 + \ldots + a_6 = 3$,

$$\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i_2'}\eta_{i_3'}\eta_{i_4'} = \eta_{i_1}^{a_1}\eta_{i_2}^{1+a_2}\eta_{i_3}^{1+a_3}\eta_{i_4}^{1+a_4}\eta_{j_2}^{a_5}\eta_{j_3}^{a_6}$$

and for some integers satisfying $0 \leq b_1, b_2, b_3 \leq 1$, and $b_1 + b_2 + b_3 = 1$,

$$W_{i_1i_4}^2 W_{i_2j_2}W_{i_3j_3}W_{i_1'i_4'}^2 W_{i_2'j_2'}W_{i_3'j_3'} = W_{i_1i_4}^{b_1+3}W_{i_2j_2}^{b_2+2}W_{i_3j_3}^{b_3+2}.$$

Similarly, by $v \sim \|\theta\|_1^2$, $0 < \eta_i \leq C\theta_i$, and uniformly for all b_1, b_2, b_3 above,

$$0 < \mathbb{E}[W_{i_1i_4}^{b_1+3}W_{i_2j_2}^{b_2+2}W_{i_3j_3}^{b_3+2}] \leq C\Omega_{i_1i_4}\Omega_{i_2j_2}\Omega_{i_3j_3} \leq C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\theta_{j_2}\theta_{j_3}.$$

Therefore under both the null and the alternative, the contribution of Case A to the variance is

(165)
$$\leq C(\|\theta\|_1)^{-6} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2,j_3\\ j_2\neq i_2, j_3\neq i_3, (j_2,j_3)\neq(i_3,i_2)}} [\sum_a \theta_{i_1}^{a_1+1}\theta_{i_2}^{a_2+2}\theta_{i_3}^{a_3+2}\theta_{i_4}^{a_4+2}\theta_{j_2}^{a_5+1}\theta_{j_3}^{a_6+1}],$$

where $a = (a_1, a_2, \ldots, a_6)$ and a_i satisfies the above constraints. Note that the right hand size

$$\leq C(\|\theta\|_1)^{-6} \cdot \max\{\|\theta\|_1^3\|\theta\|_3^9, \|\theta\|_1^2\|\theta\|^4\|\theta\|_3^6, \|\theta\|_1\|\theta\|^8\|\theta\|_3^3, \|\theta\|^{12}\} \leq C\|\theta\|_3^9/\|\theta\|_1^3.$$

Here in the last inequality we have used $\|\theta\|^2 \leq \sqrt{\|\theta\|_1\|\theta\|_3^3}$.

Consider Case B. In this case, $\{i_2, i_3, j_2, j_3\} = \{i_2', i_3', j_2', j_3'\}$, and for some integers $0 \leq c_1, c_2, c_3, c_4 \leq 1$, $c_1 + c_2 + c_3 + c_4 = 2$,

$$\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i_2'}\eta_{i_3'}\eta_{i_4'} = \eta_{i_2}^{c_1+1}\eta_{i_3}^{c_2+1}\eta_{i_4}\eta_{j_2}^{c_3}\eta_{j_3}^{c_4}\eta_{i_4'},$$

and

$$W_{i_1i_4}^2 W_{i_2j_2}W_{i_3j_3}W_{i_1'i_4'}^2 W_{i_2'j_2'}W_{i_3'j_3'} = W_{i_1i_4}^2 W_{i_2j_2}^2 W_{i_3j_3}^2 W_{i_1'i_4'}^2,$$

where the four W terms on the right are independent of each other. Similarly, by $v \sim \|\theta\|_1^2$, $0 < \eta_i \leq C\theta_i$,

$$0 < \mathbb{E}[W_{i_1i_4}^2 W_{i_2j_2}^2 W_{i_3j_3}^2 W_{i_1'i_4'}^2] \leq C\Omega_{i_1i_4}\Omega_{i_2j_2}\Omega_{i_3j_3}\Omega_{i_1'i_4'} \leq C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\theta_{j_2}\theta_{j_3}\theta_{i_1'}\theta_{i_4'},$$

we have that under both the null and the alternative, the contribution of Case B to the variance

$$\leq C(\|\theta\|_1)^{-6} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i_1',i_4'(dist)}} \sum_{\substack{j_2,j_3\\ (j_2,j_3)\neq(i_3,i_2)}} \theta_{i_1}\theta_{i_2}^{c_1+2}\theta_{i_3}^{c_2+2}\theta_{i_4}^2\theta_{j_2}^{c_3+1}\theta_{j_3}^{c_4+1}\theta_{i_1'}\theta_{i_4'}^2,$$

where the right hand size

(166)

$$\leq C(\|\theta\|_1)^{-6}\cdot\|\theta\|_1^2\|\theta\|^4\cdot\max\{\|\theta\|_1^2\|\theta\|_3^6, \|\theta\|_1\|\theta\|^4\|\theta\|_3^3, \|\theta\|^8\} \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2.$$

Here we have again used $\|\theta\|^2 \leq \sqrt{\|\theta\|_1\|\theta\|_3^3}$.

Finally, combining (165) and (166) gives

$$\mathbb{E}[X_{b1}^2] \leq C(\|\theta\|_3^9/\|\theta\|_1^3 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2) \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2,$$

which proves (160).

Consider $\mathbb{E}[X_c^2]$. Consider the terms in the sum,

$$\eta_{i_2}\eta_{i_3}\eta_{i_4}W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i_1i_4}, \qquad \text{and} \qquad \eta_{i_2'}\eta_{i_3'}\eta_{i_4'}W_{i_1'j_1'}W_{i_2'j_2'}W_{i_3'j_3'}W_{i_1'i_4'}.$$

Each term has a mean 0, and two terms are uncorrelated with each other if only if the two sets of random variables $\{W_{i_1j_1}, W_{i_2j_2}, W_{i_3j_3}, W_{i_1i_4}\}$ and $\{W_{i_1'j_1'}, W_{i_2'j_2'}, W_{i_3'j_3'}, W_{i_1'i_4'}\}$ are identical (however, it is possible that $W_{i_1j_1}$ does not equal to $W_{i_1j_1'}$ but equals to $W_{i_2'j_2'}$, say). Additionally, the indices $i_2', i_3', i_4' \in \{i_1, i_2, i_3, i_4, j_1, j_2, j_3\}$, and it follows

$$\mathbb{E}[X_c^2] \leq Cv^{-3} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,j_2,j_3\\ j_1\notin\{i_1,i_4\},(j_1,j_3)\neq(i_3,i_1)\\ (j_2,j_3)\neq(i_3,i_2),(j_2,j_1)\neq(i_2,i_1)}} [\sum_a \eta_{i_1}^{a_1}\eta_{i_2}^{a_2+1}\eta_{i_3}^{a_3+1}\eta_{i_4}^{a_4+1}\eta_{j_1}^{a_5}\eta_{j_2}^{a_6}\eta_{j_3}^{a_7}]\cdot\mathbb{E}[W_{i_1j_1}^2W_{i_2j_2}^2W_{i_3j_3}^2W_{i_1j_1}^2],$$

where $a = (a_1, a_2, \cdots, a_7)$ and the power $0 \leq a_1, a_2, \cdots, a_7 \leq 1$, and $a_1 + a_2 + \cdots + a_7 = 3$. Note that $W_{i_1j_1}, W_{i_2j_2}, W_{i_3j_3}$ and $W_{i_1i_4}$ are independent and $\mathbb{E}(W_{ij}^2) \leq \Omega_{ij} \leq C\theta_i\theta_j$, $1 \leq i, j \leq n$, $i \neq j$,

$$\mathbb{E}[W_{i_1j_1}^2W_{i_2j_2}^2W_{i_3j_3}^2W_{i_1i_4}^2] \leq \Omega_{i_1j_1}\Omega_{i_2j_2}\Omega_{i_3j_3}\Omega_{i_1i_4} \leq C\theta_{i_1}^2\theta_{i_2}\theta_{i_3}\theta_{i_4}\theta_{j_1}\theta_{j_2}\theta_{j_3}.$$

Also, recall that both under the null and the alternative, $v \asymp \|\theta\|_1^2$ and $0 < \eta_i \leq C\theta_i$, $1 \leq i \leq n$. Combining these gives

$$\mathbb{E}[X_c^2] \leq C(\|\theta\|_1)^{-6} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,j_2,j_3\\ j_1\notin\{i_1,i_4\},(j_1,j_3)\neq(i_3,i_1)\\ (j_2,j_3)\neq(i_3,i_2),(j_2,j_1)\neq(i_2,i_1)}} [\sum_a \eta_{i_1}^{a_1+2}\eta_{i_2}^{a_2+2}\eta_{i_3}^{a_3+2}\eta_{i_4}^{a_4+2}\eta_{j_1}^{a_5+1}\eta_{j_2}^{a_6+1}\eta_{j_3}^{a_7+1}],$$

where the last term

$$\leq C\sum_a \|\theta\|_{a_1+2}^{a_1+2}\cdot\|\theta\|_{a_2+2}^{a_2+2}\cdot\|\theta\|_{a_3+2}^{a_3+2}\cdot\|\theta\|_{a_4+2}^{a_4+2}\|\theta\|_{a_5+1}^{a_5+1}\|\theta\|_{a_6+1}^{a_6+1}\|\theta\|_{a_7+1}^{a_7+1}/\|\theta\|_1^6.$$

Since $a_1, a_2, \cdots, a_7$ have to take values from $\{0, 1\}$ and their sum is 3, the above term

$$\leq C\|\theta\|^2\|\theta\|_3^9/\|\theta\|_1^3 = o(\|\theta\|_3^3),$$

where we have used $\|\theta\|_3^3 \ll \|\theta\|_2^2 \ll \|\theta\|_1$. Combining these gives

(167) $$\mathbb{E}[X_c^2] \leq C\|\theta\|^2\|\theta\|_3^9/\|\theta\|_1^3.$$

Finally, inserting (159), (164), and (167) into (148) gives that both under the null and the alternative,

$$\mathrm{Var}(T_{11}) \le C[\|\theta\|^8/\|\theta\|_1^4 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2 + \|\theta\|^2\|\theta\|_3^9/\|\theta\|_1^3] \le C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2,$$

where we have used $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$ and $\|\theta\|_3^3/\|\theta\|_1 = o(1)$. This gives (133) and completes the proof for Item (a).

Consider Item (b). The goal is to show (134). Recall that

$$T_{1b} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\big[(\eta_{i_1}-\tilde{\eta}_{i_1})(\eta_{i_2}-\tilde{\eta}_{i_2})(\eta_{i_4}-\tilde{\eta}_{i_4})\big]\cdot W_{i_4 i_1},$$

and that

$$\widetilde{\eta} - \eta = v^{-1/2}W1_n.$$

Plugging this into T_{1b} gives

$$\begin{aligned} T_{1b} &= -v^{-3/2}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\Big(\sum_{j_1\neq i_1} W_{i_1j_1}\Big)\Big(\sum_{j_2\neq i_2} W_{i_2j_2}\Big)\Big(\sum_{j_4\neq i_4} W_{i_4j_4}\Big)W_{i_1i_4} \\ &= -\frac{1}{v^{3/2}}\sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_1\neq i_1,j_2\neq i_2,j_4\neq i_4}} \eta_{i_2}\eta_{i_3}^2 W_{i_1j_1}W_{i_2j_2}W_{i_4j_4}W_{i_1i_4}. \end{aligned}$$

By basic combinatorics and careful observations, we have

$$W_{i_1j_1}W_{i_2j_2}W_{i_4j_4}W_{i_1i_4} = \begin{cases} W_{i_1i_4}^3 W_{i_2j_2}, & \text{if } j_1 = i_4, j_4 = i_1, \\ W_{i_1i_2}^2 W_{i_1i_4}^2, & \text{if } j_1 = i_2, j_2 = i_1, j_4 = i_1, \\ W_{i_1i_4}^2 W_{i_2i_4}^2, & \text{if } j_1 = i_4, j_2 = i_4, j_4 = i_2, \\ W_{i_1i_2}^2 W_{i_4j_4} W_{i_1i_4}, & \text{if } j_1 = i_2, j_2 = i_1, \\ W_{i_1i_4}^2 W_{i_1j_1} W_{i_2j_2}, & \text{if } j_4 = i_1, \\ W_{i_1i_4}^2 W_{i_2j_2} W_{i_4j_4}, & \text{if } j_1 = i_4, \{i_2,j_2\}\neq\{i_4,j_4\}, \\ W_{i_2i_4}^2 W_{i_1j_1} W_{i_1i_4}, & \text{if } j_2 = i_4, j_4 = i_2, \\ W_{i_1j_1}W_{i_2j_2}W_{i_4j_4}W_{i_1i_4}, & \text{otherwise.} \end{cases} \tag{168}$$

This allows us to further split T_{1b} into 8 different terms:

$$T_{1b} = Y_{a1} + Y_{a2} + Y_{a3} + Y_{b1} + Y_{b2} + Y_{b3} + Y_{b4} + Y_c, \tag{169}$$

where

$$Y_{a1} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{j_2(j_2\neq i_2)} \eta_{i_2}\eta_{i_3}^2 W_{i_1i_4}^3 W_{i_2j_2},$$

$$Y_{a2} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2 W_{i_1i_2}^2 W_{i_1i_4}^2,$$

$$Y_{a3} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2 W_{i_1i_4}^2 W_{i_2i_4}^2,$$

$$Y_{b1} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{j_4(j_4\neq i_4)} \eta_{i_2}\eta_{i_3}^2 W_{i_1i_2}^2 W_{i_4j_4}W_{i_1i_4},$$

$$Y_{b2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_1 (j_1 \neq i_1), j_2 (j_2 \neq i_2) \\ \{i_1,j_1\} \neq \{i_2,j_2\}}} \eta_{i_2} \eta_{i_3}^2 W_{i_1 i_4}^2 W_{i_1 j_1} W_{i_2 j_2},$$

$$Y_{b3} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_2 (j_2 \neq i_2), j_4 (j_4 \neq i_4) \\ \{i_2,j_2\} \neq \{i_4,j_4\}}} \eta_{i_2} \eta_{i_3}^2 W_{i_1 i_4}^2 W_{i_2 j_2} W_{i_4 j_4},$$

$$Y_{b4} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{j_1 (j_1 \neq i_1)} \eta_{i_2} \eta_{i_3}^2 W_{i_2 i_4}^2 W_{i_1 j_1} W_{i_1 i_4},$$

$$Y_c = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_1, j_2, j_4 \\ j_1 \notin \{i_2,i_4\}, j_2 \notin \{i_1,i_4\}, j_4 \notin \{i_1,i_2\}}} \eta_{i_2} \eta_{i_3}^2 W_{i_1 j_1} W_{i_2 j_2} W_{i_4 j_4} W_{i_1 i_4}.$$

We now show the two claims in (134) separately.

Consider the first claim of (134). It is seen that out of the 8 terms on the right hand side of (196), the mean of all terms are 0, except that of the Y_{a2} and Y_{a3}. Note that for any $1 \leq i, j \leq n$, $i \neq j$, $\mathbb{E}[W_{ij}^2] = \Omega_{ij}(1 - \Omega_{ij})$, where Ω_{ij} are upper bounded by $o(1)$ uniformly for all such i, j. It follows

$$\begin{aligned} \mathbb{E}[Y_{a2}] &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \eta_{i_2} \eta_{i_3}^2 \mathbb{E}[W_{i_1 i_2}^2] \mathbb{E}[W_{i_1 i_4}^2] \\ &= -(1 + o(1)) \cdot v^{-3/2} \sum_{i_1,i_2,i_3,i_4 (dist)} \eta_{i_2} \eta_{i_3}^2 \Omega_{i_1 i_2} \Omega_{i_1 i_4}. \end{aligned}$$

Since for any $1 \leq i, j \leq n$, $i \neq j$, $0 < \eta_i \leq C\theta_i$, $\Omega_{ij} \leq C\theta_i\theta_j$ and $v \asymp \|\theta\|_1^2$,

$$|\mathbb{E}[Y_{a2}]| \leq C(\|\theta\|_1)^{-3} \sum_{i_1,i_2,i_3,i_4 (dist)} \theta_{i_1}^2 \theta_{i_2}^2 \theta_{i_3}^2 \theta_{i_4} \leq C\|\theta\|^6 / \|\theta\|_1^2.$$

Therefore,

$$|\mathbb{E}[Y_{a2}]| \leq C\|\theta\|^6 / \|\theta\|_1^2. \tag{170}$$

By symmetry, we similarly find

$$|\mathbb{E}[Y_{a3}]| \leq C\|\theta\|^6 / \|\theta\|_1^2. \tag{171}$$

Combining (170) and (171) gives

$$\mathbb{E}[|T_{1b}|] \leq C\|\theta\|^6 / \|\theta\|_1^2.$$

This completes the proof of the first claim of (134).

We now show the second claim of (134) . By Cauchy-Schwartz inequality,

$$\begin{aligned} \mathrm{Var}(T_{1b}) &\leq C(\mathrm{Var}(Y_{a1}) + \mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3}) + \sum_{s=1}^{4} \mathrm{Var}(Y_{bs}) + \mathrm{Var}(Y_c)) \\ &\leq C(\mathrm{Var}(Y_{a1}) + \mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3}) + \sum_{s=1}^{4} \mathbb{E}[Y_{bs}^2] + \mathbb{E}[Y_c^2]). \quad (172) \end{aligned}$$

We now show $\mathrm{Var}(Y_{a1})$, $\mathrm{Var}(Y_{a2})$, $\mathrm{Var}(Y_{a3})$, $\sum_{s=1}^{4} \mathbb{E}[Y_{bs}^2]$, and $\mathbb{E}[Y_c^2]$, separately.

Consider $\mathrm{Var}(Y_{a1})$. Recalling $\mathbb{E}[Y_{a1}] = 0$, we write $\mathrm{Var}(Y_{a1})$ as

$$v^{-3} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i_1',i_2',i_3',i_4'(dist)}} \sum_{j_2(j_2\neq i_2)} \sum_{j_2'(j_2'\neq i_2')} \eta_{i_2}\eta_{i_3}^2\eta_{i_2'}\eta_{i_3'}^2 \mathbb{E}\big[W_{i_1i_4}^3 W_{i_2j_2} W_{i_1'i_4'}^3 W_{i_2'j_2'}\big]. \tag{173}$$

In the sum, a term is nonzero only when one of the following cases happens.

- (A). $\{W_{i_1i_4}, W_{i_2j_2}, W_{i_1'i_4'}, W_{i_2'j_2'}\}$ has 2 distinct random variables.
- (B). $\{W_{i_1i_4}, W_{i_2j_2}, W_{i_1'i_4'}, W_{i_2'j_2'}\}$ has 3 distinct random variables. While it may seem we have 4 possibilities in this case, but the only one that has a nonzero mean is when $W_{i_2j_2} = W_{i_2'j_2'}$.

For Case (A), the two sets $\{i_1, i_2, i_4, j_2\}$ and $\{i_1', i_2', i_4', j_2'\}$ are identical, and so for two integers $0 \le b_1, b_2 \le 1$ and $b_1 + b_2 = 1$,

$$W_{i_1i_4}^3 W_{i_2j_2} W_{i_1'i_4'}^3 W_{i_2'j_2'} = W_{i_1i_4}^{4+2b_1} W_{i_2j_2}^{2+2b_2},$$

and so

$$\mathbb{E}[W_{i_1i_4}^3 W_{i_2j_2} W_{i_1'i_4'}^3 W_{i_2'j_2'}] = \mathbb{E}[W_{i_1i_4}^{4+2b_1} W_{i_2j_2}^{2+2b_2}] = \mathbb{E}[W_{i_1i_4}^{4+2b_1}]\mathbb{E}[W_{i_2j_2}^{2+2b_2}],$$

Note that for any integer $2 \le b \le 6$,

$$0 < \mathbb{E}[W_{ij}^b] \le C\Omega_{ij},$$

where note that $\Omega_{ij} \le C\theta_i\theta_j$ for all $1 \le i, j \le n$, $i \le j$. Recall that $v \sim \|\theta\|_1^2$, and that $0 < \eta_i \le C\theta_i$ for all $1 \le i \le n$. Combining these that, the contribution of Case (A) to $\mathrm{Var}(Y_{a1})$ is no more than

$$C(\|\theta\|_1)^{-6} \sum_{i_1,\cdots,i_4(dist)} \sum_{i_3',j_2} \sum_{a} \theta_{i_1}^{a_1+1}\theta_{i_2}^{a_2+2}\theta_{i_3}^2\theta_{i_4}^{a_3+1}\theta_{i_3'}^2\theta_{j_2}^{a_4+1}, \tag{174}$$

where $a = (a_1, a_2, a_3, a_4)$ and each a_i is either 0 and 1, satisfying $a_1 + a_2 + a_3 + a_4 = 1$. Note that the right hand side of (174) is no greater than

$$C(\|\theta\|_1)^{-6}\max\{\|\theta\|_1^3\|\theta\|^4\|\theta\|_3^3, \|\theta\|_1^2\|\theta\|^8\} \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1^3,$$

where we have used $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$.

Next, consider Case (B). In this case, $\{i_2, j_2\} = \{i_2', j_2'\}$ and

$$W_{i_1i_4}^3 W_{i_2j_2} W_{i_1'i_4'}^3 W_{i_2'j_2'} = W_{i_1i_4}^3 W_{i_2j_2}^2 W_{i_1'i_4'}^3,$$

and by similar argument,

$$0 < \mathbb{E}[W_{i_1i_4}^3 W_{i_2j_2}^2 W_{i_1'i_4'}^3] \le C\Omega_{i_1i_4}\Omega_{i_2j_2}\Omega_{i_1'i_4'}. \tag{175}$$

Recall that $\Omega_{ij} \le C\theta_i\theta_j$ for all $1 \le i, j \le n$, $i \le j$, that $v \sim \|\theta\|_1^2$, and that $0 < \eta_i \le C\theta_i$ for all $1 \le i \le n$. Combining this with (173), the contribution of this case to $\mathrm{Var}(Y_{a1})$

$$\le C(\|\theta\|_1)^{-6} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i_1',i_3',i_4'(dist)}} \sum_{j_2} C\theta_{i_1}\theta_{i_2}^{2+b_1}\theta_{i_3}^2\theta_{i_4}\theta_{i_1'}\theta_{i_3'}^2\theta_{i_4'}\theta_{j_2}^{1+b_2}, \tag{176}$$

where similarly b_1, b_2 are either 0 or 1 and $b_1 + b_2 = 1$. By similar argument, the right hand side

$$\le C\|\theta\|_1^{-6} \cdot [\|\theta\|_1^5\|\theta\|^4\|\theta\|_3^3 + \|\theta\|_1^4\|\theta\|^8] \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1,$$

where we've used Cauchy-Schwartz inequality that $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$.

Now, inserting (174) and (176) into (173) gives

$$\mathrm{Var}(Y_{a1}) \le C[\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1^3 + \|\theta\|^4\|\theta\|_3^3/\|\theta\|_1] \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1, \tag{177}$$

where we have used $\|\theta\|_1 \to \infty$ and $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$. This shows

$$\mathrm{Var}(Y_{a1}) \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1. \tag{178}$$

Next, we consider $\mathrm{Var}(Y_{a2})$ and $\mathrm{Var}(Y_{a3})$. The proofs are similar to that of $\mathrm{Var}(X_a)$ of Item (a), so we skip the detail, but claim that

$$\mathrm{Var}(Y_{a2}) \le C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \tag{179}$$

and

$$\mathrm{Var}(Y_{a3}) \le C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4. \tag{180}$$

Combining (178), (179), and (180) gives

$$\mathrm{Var}(Y_{a1})+\mathrm{Var}(Y_{a2})+\mathrm{Var}(Y_{a3}) \le C[\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1+\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4] \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1, \tag{181}$$

where we have used the universal inequality that $\|\theta\|_3^3 \le \|\theta\|_1^3$.

Next, consider $\sum_{s=1}^4 \mathbb{E}[Y_{bs}^2]$. For each $1 \le s \le 4$, the study of $\mathbb{E}[Y_{bs}^2]$ is similar to that of $\mathbb{E}[X_{b1}^2]$ in Item (a), so we skip the details. We have that both under the null and the alternative,

$$\mathbb{E}[Y_{b1}^2] \le C\|\theta\|^{12}/\|\theta\|_1^4, \tag{182}$$

$$\mathbb{E}[Y_{b2}^2] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1, \tag{183}$$

$$\mathbb{E}[Y_{b3}^2] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1, \tag{184}$$

$$\mathbb{E}[Y_{b4}^2] \le C\|\theta\|^{12}/\|\theta\|_1^4. \tag{185}$$

Therefore,

$$\sum_{s=1}^4 \mathbb{E}[Y_{bs}^2] \le C[\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^{12}/\|\theta\|_1^4] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1. \tag{186}$$

Third, we consider $\mathbb{E}[Y_c^2]$. The proof is very similar to that of $\mathbb{E}[X_c^2]$ and we have that both under the null and the alternative,

$$\mathbb{E}[Y_c^2] \le C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3. \tag{187}$$

Finally, combining (181), (186), and (187) with (172) gives

$$\mathrm{Var}(T_{1b}) \le C[\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1, \tag{188}$$

where we have used $\|\theta\| \to \infty$ and $\|\theta\|^2 \ll \|\theta\|_1$. This completes the proof of (134).

Consider Item (c). The goal is to show (135). Recall that

$$T_{1c} = \sum_{i_1,i_2,i_3,i_4 (dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\left[(\eta_{i_2}-\tilde{\eta}_{i_2})^2(\eta_{i_3}-\tilde{\eta}_{i_3})\right] \cdot W_{i_4 i_1},$$

and that

$$\widetilde{\eta} - \eta = v^{-1/2}W\mathbf{1}_n.$$

Plugging this into T_{1c} gives

$$T_{1c} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\Big(\sum_{j_2\neq i_2} W_{i_2j_2}\Big)\Big(\sum_{\ell_2\neq i_2} W_{i_2\ell_2}\Big)\Big(\sum_{j_3\neq i_3} W_{i_3j_3}\Big)W_{i_1i_4}$$
$$= -\frac{1}{v^{3/2}} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_2\neq i_2,\ell_2\neq i_2,j_3\neq i_3}} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}W_{i_1i_4}.$$

By basic combinatorics and careful observations, we have

$$W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}W_{i_1i_4} = \begin{cases} W_{i_2i_3}^3 W_{i_1i_4}, & \text{if } j_2=\ell_2=i_3, j_3=i_2,\\ W_{i_2j_2}^2 W_{i_3j_3}W_{i_1i_4}, & \text{if } j_2=\ell_2, (j_3,j_2)\neq(i_2,i_3),\\ W_{i_2i_3}^2 W_{i_2\ell_2}W_{i_1i_4}, & \text{if } j_2=i_3, j_3=i_2, \ell_2\neq i_3,\\ W_{i_2i_3}^2 W_{i_2j_2}W_{i_1i_4}, & \text{if } \ell_2=i_3, j_3=i_2, j_2\neq i_3,\\ W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}W_{i_1i_4}, & \text{otherwise.} \end{cases} \tag{189}$$

This allows us to further split T_{1c} into 5 different terms:

$$T_{1c} = Z_a + Z_{b1} + Z_{b2} + Z_{b3} + Z_c, \tag{190}$$

where

$$Z_a = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2i_3}^3 W_{i_1i_4},$$

$$Z_{b1} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2,(j_3,j_2)\neq(i_2,i_3)} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2j_2}^2 W_{i_3j_3}W_{i_1i_4},$$

$$Z_{b2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2=i_3,j_3=i_2\\ \ell_2\neq i_3}} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2i_3}^2 W_{i_2\ell_2}W_{i_1i_4},$$

$$Z_{b3} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{\ell_2=i_3,j_3=i_2\\ j_2\neq i_3}} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2i_3}^2 W_{i_2j_2}W_{i_1i_4},$$

$$Z_c = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2,\ell_2,j_3\\ j_2\neq\ell_2,j_2,\ell_2\neq i_3,j_3\neq i_2}} \eta_{i_1}\eta_{i_3}\eta_{i_4} W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}W_{i_1i_4}.$$

We now show the two claims in (135) separately. The proof of the first claim is trivial, so we only show the second claim of (135).

Consider the second claim of (135). By Cauchy-Schwartz inequality,

$$\mathrm{Var}(T_{1c}) \leq C(\mathrm{Var}(Z_a) + \mathrm{Var}(Z_{b1}) + \mathrm{Var}(Z_{b2}) + \mathrm{Var}(Z_{b3}) + \mathrm{Var}(Z_c))$$
$$\leq C(\mathbb{E}[Z_a^2] + \sum_{s=1}^{3}\mathbb{E}[Z_{bs}^2] + \mathbb{E}[Z_c^2]). \tag{191}$$

Note that

- The proof of $\mathrm{Var}(Z_a)$ is similar to that of $\mathrm{Var}(Y_a)$ in Item (b).
- The proof of $\sum_{s=1}^{3}\mathbb{E}[Z_{bs}^2]$ is similar to that of $\sum_{s=1}^{4}\mathbb{E}[X_{bs}^2]$ in Item (a).
- The proof of $\mathbb{E}[Z_c^2]$ is similar to that of $\mathbb{E}[X_c^2]$ in Item (a).

For these reasons, we omit the proof details and only state the claims. We have that under both the null and the alternative,

$$\mathrm{Var}(Z_a) \le C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \tag{192}$$

$$\sum_{s=1}^{3}\mathbb{E}[Z_{bs}^2] \le C\|\theta\|_3^9/\|\theta\|_1, \tag{193}$$

and

$$\mathbb{E}[Z_c^2] \le C\|\theta\|^2\|\theta\|_3^9/\|\theta\|_1^3. \tag{194}$$

Inserting (192), (193), and (194) into (191) gives

$$\mathrm{Var}(T_{1c}) \le C[\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4 + \|\theta\|_3^9/\|\theta\|_1 + \|\theta\|^2\|\theta\|_3^9/\|\theta\|_1^3] \le C\|\theta\|_3^9/\|\theta\|_1,$$

where we have used $\|\theta\|_3^3 \ll \|\theta\|^2 \ll \|\theta\|_1$, $\|\theta\|^4 \le \|\theta\|_1\|\theta\|_3^3$ and $\|\theta\|_1 \to \infty$. This proves (135).

Consider Item (d). The goal is to show (136) and (137). Recall that

$$T_{1d} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}^2\left[(\eta_{i_2}-\tilde{\eta}_{i_2})^2(\eta_{i_4}-\tilde{\eta}_{i_4})\right]\cdot W_{i_4 i_1}.$$

and that

$$\widetilde{\eta} - \eta = v^{-1/2}W1_n.$$

Plugging this into T_{1d} gives

$$\begin{aligned} T_{1d} &= -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}^2\Big(\sum_{j_2\neq i_2} W_{i_2 j_2}\Big)\Big(\sum_{\ell_2\neq i_2} W_{i_2\ell_2}\Big)\Big(\sum_{j_4\neq i_4} W_{i_4 j_4}\Big)W_{i_1 i_4} \\ &= -\frac{1}{v^{3/2}}\sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_2\neq i_2,\ell_2\neq i_2,j_4\neq i_4}} \eta_{i_1}\eta_{i_3}^2 W_{i_2 j_2}W_{i_2\ell_2}W_{i_4 j_4}W_{i_1 i_4}. \end{aligned}$$

By basic combinatorics and careful observations, we have

$$W_{i_2j_2}W_{i_2\ell_2}W_{i_4j_4}W_{i_1i_4} = \begin{cases} W_{i_2i_4}^3 W_{i_1i_4}, & \text{if } j_2=\ell_2=i_4, j_4=i_2,\\ W_{i_2j_2}^2 W_{i_1i_4}^2, & \text{if } j_2=\ell_2, j_4=i_1,\\ W_{i_2j_2}^2 W_{i_4j_4}W_{i_1i_4}, & \text{if } j_2=\ell_2, j_4\neq i_1, (j_2,j_4)\neq(i_4,i_2),\\ W_{i_2j_2}W_{i_2i_4}^2 W_{i_1i_4}, & \text{if } \ell_2=i_4, j_4=i_2, j_2\neq i_4,\\ W_{i_2\ell_2}W_{i_2i_4}^2 W_{i_1i_4}, & \text{if } j_2=i_4, j_4=i_2, \ell_2\neq i_4,\\ W_{i_2j_2}W_{i_2\ell_2}W_{i_1i_4}^2, & \text{if } j_4=i_1, j_2\neq\ell_2,\\ W_{i_2j_2}W_{i_2\ell_2}W_{i_4j_4}W_{i_1i_4}, & \text{otherwise.} \end{cases} \tag{195}$$

This allows us to further split T_{14} into 7 different terms:

$$T_{1d} = U_{a1} + U_{a2} + U_{b1} + U_{b2} + U_{b3} + U_{b4} + U_c, \tag{196}$$

where

$$U_{a1} = -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}^2 W_{i_2i_4}^3 W_{i_1i_4},$$

$$U_{a2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2} \eta_{i_1}\eta_{i_3}^2 W_{i_2j_2}^2 W_{i_1i_4}^2,$$

$$U_{b1} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2(j_2\neq i_2),j_4(j_4\neq i_4)\\ j_4\neq i_1,(j_2,j_4)\neq(i_4,i_2)}} \eta_{i_1}\eta_{i_3}^2 W_{i_2j_2}^2 W_{i_4j_4} W_{i_1i_4},$$

$$U_{b2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2(j_2\neq i_4)} \eta_{i_1}\eta_{i_3}^2 W_{i_2j_2} W_{i_2i_4}^2 W_{i_1i_4},$$

$$U_{b3} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\ell_2(\ell_2\neq i_4)} \eta_{i_1}\eta_{i_3}^2 W_{i_2\ell_2} W_{i_2i_4}^2 W_{i_1i_4},$$

$$U_{b4} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2\neq\ell_2} \eta_{i_1}\eta_{i_3}^2 W_{i_2j_2} W_{i_2\ell_2} W_{i_1i_4}^2,$$

$$U_c = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2,\ell_2,j_4,W\text{dist}} \eta_{i_1}\eta_{i_3}^2 W_{i_2j_2} W_{i_2\ell_2} W_{i_4j_4} W_{i_1i_4}.$$

We now show (136) and (137) separately.

Consider (136). It is seen that out of the 7 terms on the right hand side of (190), all terms are mean 0, except for the second term U_{a2}. Note that for any $1 \le i,j \le n$, $i \neq j$, $\mathbb{E}[W_{ij}^2] = \Omega_{ij}(1-\Omega_{ij})$, where Ω_{ij} are upper bounded by $o(1)$ uniformly for all such i,j. It follows

$$\begin{aligned}\mathbb{E}[U_{a2}] &= -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2} \eta_{i_1}\eta_{i_3}^2 \mathbb{E}[W_{i_2j_2}^2]\mathbb{E}[W_{i_1i_4}^2] \\ &= -(1+o(1)) \cdot v^{-3/2} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_2}} \eta_{i_1}\eta_{i_3}^2 \Omega_{i_2j_2}\Omega_{i_1i_4}.\end{aligned}$$

Under null, for any $1 \le i,j \le n$, $i \neq j$, $\eta_i = (1+o(1))\theta_i$, $\Omega_{ij} = (1+o(1))\theta_i\theta_j$ and $v \asymp \|\theta\|_1^2$,

$$\mathbb{E}[U_{a2}] = (\|\theta\|_1)^{-3} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2} \theta_{i_1}^2\theta_{i_2}\theta_{i_3}^2\theta_{i_4}\theta_{j_2} = -(1+o(1))\|\theta\|^4,$$

and under alternative, a similar arguments yields

$$|\mathbb{E}[U_{a1}]| \le C\|\theta\|^4. \tag{197}$$

This proves (136).

We now consider (137). By Cauchy-Schwartz inequality,

$$\begin{aligned}\mathrm{Var}(T_{1d}) &\le C(\mathrm{Var}(U_{a1}) + \mathrm{Var}(U_{a2}) + \sum_{s=1}^{4}\mathrm{Var}(U_{bs}) + \mathrm{Var}(U_c)) \\ &\le C(\mathrm{Var}(U_{a1}) + \mathrm{Var}(U_{a2}) + \sum_{s=1}^{4}\mathbb{E}[U_{bs}^2] + \mathbb{E}[U_c^2]).\end{aligned} \tag{198}$$

Note that

- The proof of U_{a1} is similar to that of Y_{a1} in Item (b).
- The proof of U_{a2} is similar to that of X_{a1} in Item (a).
- The proof of U_{bs}, $1 \le s \le 4$, is similar to that of X_{b1} in Item (a).

- The proof of U_c is similar to that of X_c in Item (a).

For these reasons, we omit the proof details, and claim that

$$\mathrm{Var}(U_{a1}) \le C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \tag{199}$$

$$\mathrm{Var}(U_{a2}) \le C\|\theta\|^4\|\theta\|_3^3/\|\theta\|_1, \tag{200}$$

$$\sum_{s=1}^{4}\mathbb{E}[U_{bs}^2] \le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1, \tag{201}$$

and

$$\mathrm{Var}(U_c) \le C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3, \tag{202}$$

Inserting (199), (200), (201), and (202) into (198) gives

$$\mathrm{Var}(T_{1d}) \le C[\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4 + \|\theta\|^4\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^6\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^3] \tag{203}$$

$$\le C\|\theta\|^6\|\theta\|_3^3/\|\theta\|_1, \tag{204}$$

where we have used $\|\theta\| \to \infty$ and $\|\theta\|_3^3 \le \|\theta\|_1^3$. This proves (137).

We now consider Item (e) and Item (f). Since the proof is similar, we only prove Item (e). The goal is to show (138). Recall that

$$T_{2a} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1}-\tilde{\eta}_{i_1})(\eta_{i_2}-\tilde{\eta}_{i_2})(\eta_{i_3}-\tilde{\eta}_{i_3})\big]\cdot\widetilde{\Omega}_{i_4 i_1}, \tag{205}$$

and

$$\widetilde{\eta} - \eta = v^{-1/2}W1_n. \tag{206}$$

Plugging (206) into (205) gives

$$\begin{aligned} T_{2a} &= -\frac{1}{v^{3/2}}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}\eta_{i_4}\Big(\sum_{j_1\neq i_1} W_{i_1j_1}\Big)\Big(\sum_{j_2\neq i_2} W_{i_2j_2}\Big)\Big(\sum_{j_3\neq i_3} W_{i_3j_3}\Big)\widetilde{\Omega}_{i_4i_1} \\ &= -\frac{1}{v^{3/2}}\sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_1\neq i_1, j_2\neq i_2, j_3\neq i_3}} \eta_{i_2}\eta_{i_3}\eta_{i_4}W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}\widetilde{\Omega}_{i_1i_4}. \end{aligned}$$

By basic combinatorics and careful observations, we have

$$W_{i_1j_1}W_{i_2j_2}W_{i_3j_3} = \begin{cases} W_{i_1i_2}^2 W_{i_3j_3}, & \text{if } j_1 = i_2, j_2 = i_1, \\ W_{i_1i_3}^2 W_{i_2j_2}, & \text{if } j_1 = i_3, j_3 = i_1, \\ W_{i_2i_3}^2 W_{i_1j_1}, & \text{if } j_2 = i_3, j_3 = i_2, \\ W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}, & \text{otherwise.} \end{cases} \tag{207}$$

This allows us to further split T_{2a} into 4 different terms:

$$T_{2a} = X_{a1} + X_{a2} + X_{a3} + X_b, \tag{208}$$

where

$$X_{a1} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_3 \neq i_3} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1i_2}^2 W_{i_3j_3} \widetilde{\Omega}_{i_1i_4},$$

$$X_{a2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2 \neq i_2} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1i_3}^2 W_{i_2j_2} \widetilde{\Omega}_{i_1i_4},$$

$$X_{a3} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_1 \neq i_1} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_2i_3}^2 W_{i_1j_1} \widetilde{\Omega}_{i_1i_4},$$

$$X_b = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,j_2,j_3 \\ j_k \neq i_\ell, k,\ell=1,2,3}} \eta_{i_2}\eta_{i_3}\eta_{i_4} W_{i_1j_1} W_{i_2j_2} W_{i_3j_3} \widetilde{\Omega}_{i_1i_4}.$$

We now consider the two claims of (138) separately. Since the mean of $X_{a1}, X_{a2}, X_{a3}, X_b$ are all 0, the first claim of (138) follows trivially, so all remains to show is the second claim of (138).

We now consider the second claim of (138). By Cauchy-Schwartz inequality,

$$\begin{aligned} \text{Var}(T_{2a}) &\le C\text{Var}(X_{a1}) + \text{Var}(X_{a2}) + \text{Var}(X_{a3}) + \text{Var}(X_b)) \\ &\le C(\mathbb{E}[X_{a1}^2] + \mathbb{E}[X_{a2}^2] + \mathbb{E}[X_{a3}^2] + \mathbb{E}[X_b^2]). \end{aligned} \tag{209}$$

We now consider $\mathbb{E}[X_{a1}^2] + \mathbb{E}[X_{a2}^2] + \mathbb{E}[X_{a3}^2]$, and $\mathbb{E}[X_b^2]$, separately.

Consider $\mathbb{E}[X_{a1}^2] + \mathbb{E}[X_{a2}^2] + \mathbb{E}[X_{a3}^2]$. We claim that both under the null and the alternative,

$$\mathbb{E}[X_{a1}^2] \le C\alpha^2 \|\theta\|^{12} \|\theta\|_3^3 / \|\theta\|_1^5, \tag{210}$$

$$\mathbb{E}[X_{a2}^2] \le C\alpha^2 \|\theta\|^{12} \|\theta\|_3^3 / \|\theta\|_1^5, \tag{211}$$

$$\mathbb{E}[X_{a3}^2] \le C\alpha^2 \|\theta\|^{12} \|\theta\|_3^3 / \|\theta\|_1^5. \tag{212}$$

Combining these gives that both under the null and the alternative,

$$\mathbb{E}[X_{a1}^2] + \mathbb{E}[X_{a2}^2] + \mathbb{E}[X_{a3}^2] \le C\alpha^2 \|\theta\|^{12} \|\theta\|_3^3 / \|\theta\|_1^5. \tag{213}$$

It remains to show (210)-(212). Since the proofs are similar, we only prove (210). Write

$$\mathbb{E}[X_{a1}^2] = v^{-3} \sum_{\substack{i_1,i_2,i_3,i_4(dist) \\ i_1',i_2',i_3',i_4'(dist)}} \sum_{\substack{j_3,j_3' \\ j_3 \neq i_3, j_3' \neq i_3'}} \eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i_2'}\eta_{i_3'}\eta_{i_4'} \mathbb{E}[W_{i_1i_2}^2 W_{i_3j_3} W_{i_1'i_2'}^2 W_{i_3'j_3'}] \widetilde{\Omega}_{i_1i_4} \widetilde{\Omega}_{i_1'i_4'}.$$

Consider the term

$$W_{i_1i_2}^2 W_{i_3j_3} W_{i_1'i_2'}^2 W_{i_3'j_3'}.$$

In order for the mean is nonzero, we have three cases

- Case A. $W_{i_1i_2} = W_{i_3'j_3'}$ and $W_{i_3j_3} = W_{i_1'i_2'}$.
- Case B. $W_{i_3j_3} = W_{i_3'j_3'}$ and $W_{i_1i_2} = W_{i_1'i_2'}$.
- Case C. $W_{i_3j_3} = W_{i_3'j_3'}$ and $W_{i_1i_2} \neq W_{i_1'i_2'}$.

Consider Case A. In this case, $\{i_1', i_2', i_3'\}$ are three distinct indices in $\{i_1, i_2, i_3, j_3\}$. In this case,

$$W_{i_1i_2}^2 W_{i_3j_3} W_{i_1'i_2'}^2 W_{i_3'j_3'} = W_{i_1i_2}^3 W_{i_3j_3}^3,$$

where by similar arguments as before

$$0 < \mathbb{E}[W_{i_1i_2}^3 W_{i_3j_3}^3] \le C\Omega_{i_1i_2}\Omega_{i_3j_3} \le C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{j_3}.$$

At the same time, recall that that $0 < \eta_i \leq C\theta_i$ for any $1 \leq i \leq n$, and that $|\widetilde{\Omega}_{ij}| \leq C\alpha\theta_i\theta_j$ for any $1 \leq i, j \leq n$, $i \neq j$, where $\alpha = |\lambda_2/\lambda_1|$ with λ_k being the k-th largest (in magnitude) eigenvalue of Ω, $1 \leq k \leq K$. By basic algebra,

$$|\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i'_2}\eta_{i'_3}\eta_{i'_4}\widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i'_1i'_4}| \leq C\alpha^2\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}^2\theta_{i'_1}\theta_{i'_2}\theta_{i'_3}\theta_{i'_4}^2.$$

Note that in the current case, $\{i_1, i_2\} = \{i'_3, j'_3\}$ and $\{i_3, j_3\} = \{i'_1, i'_2\}$, so for some integers $0 \leq b_1, b_2 \leq 1$ and $b_1 + b_2 = 1$,

$$\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}^2\theta_{i'_1}\theta_{i'_2}\theta_{i'_3}\theta_{i'_4}^2 = \theta_{i_1}^{1+b_1}\theta_{i_2}^{1+b_2}\theta_{i_3}^2\theta_{j_3}\theta_{i_4}^2\theta_{i'_4}^2.$$

Recall that $v \asymp \|\theta\|_1^2$. Combining these, the contribution of Case (A) to $\mathbb{E}[X_{a1}^2]$ is no greater than

$$C\alpha^2(\|\theta\|_1)^{-6}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{i'_4}\sum_{j_3(j_3\neq i_3)}\sum_{b_1,b_2(b_1+b_2=1)}\theta_{i_1}^{2+b_1}\theta_{i_2}^{2+b_2}\theta_{i_3}^3\theta_{j_3}^2\theta_{i_4}^2\theta_{i'_4}^2,$$

where the right hand side $\leq C\alpha^2 \cdot \|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6$. This shows that the contribution of Case (A) to $\mathbb{E}[X_{a1}^2]$ is no greater than

$$C\alpha^2 \cdot \|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6. \tag{214}$$

Consider Case B. By similar arguments,

$$W_{i_1i_2}^2W_{i_3j_3}W_{i'_1i'_2}^2W_{i'_3j'_3} = W_{i_1i_2}^6W_{i_3j_3}^2,$$

where

$$\mathbb{E}[W_{i_1i_2}^6W_{i_3j_3}^2] \leq C\Omega_{i_1i_2}\Omega_{i_3j_3} \leq C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{j_3},$$

Also, by similar arguments,

$$|\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i'_2}\eta_{i'_3}\eta_{i'_4}\widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i'_1i'_4}| \leq C\alpha^2\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}^2\theta_{i'_1}\theta_{i'_2}\theta_{i'_3}\theta_{i'_4}^2,$$

where as $W_{i_1i_2} = W_{i'_1i'_2}$ and $W_{i_3j_3} = W_{i'_3j'_3}$, the right hand side

$$\leq C\alpha^2\theta_{i_1}^2\theta_{i_2}^2\theta_{i_3}^{1+c_1}\theta_{j_3}^{c_2}\theta_{i_4}^2\theta_{i'_4}^2,$$

where $0 < c_1, c_2 \leq$ are integers satisfying $c_1 + c_2 = 1$. Recall $v \sim \|\theta\|_1^2$. Combining these, the contribution of Case (B) to $\mathbb{E}[X_{a1}^2]$

$$\leq C\alpha^2(\|\theta\|_1)^{-6}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{i'_4}\sum_{j_3(j_3\neq i_3)}\sum_{b_1,b_2(b_1+b_2=1)}\theta_{i_1}^3\theta_{i_2}^3\theta_{i_3}^{2+c_1}\theta_{j_3}^{1+c_2}\theta_{i_4}^2\theta_{i'_4}^2,$$

where by $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, the above term

$$\leq C\alpha^2[\|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^5, \|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6] \leq C\alpha^2\|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^5.$$

This shows that the contribution of Case (B) to $\mathbb{E}[X_{a1}^2]$ is no greater than

$$C\|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^5. \tag{215}$$

Consider Case (C). In this case,

$$W_{i_1i_2}^2W_{i_3j_3}W_{i'_1i'_2}^2W_{i'_3j'_3} = W_{i_1i_2}^2W_{i_3j_3}^2W_{i'_1i'_2}^2,$$

where by similar arguments,

$$\mathbb{E}[W_{i_1i_2}^2W_{i_3j_3}^2W_{i'_1i'_2}^2] \leq C\Omega_{i_1i_2}\Omega_{i_3j_3}\Omega_{i'_1i'_2} \leq C\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{j_3}\theta_{i'_1}\theta_{i'_2}.$$

Also, by similar arguments,

$$|\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i'_2}\eta_{i'_3}\eta_{i'_4}\widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i'_1i'_4}| \leq C\alpha^2\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}^2\theta_{i'_1}\theta_{i'_2}\theta_{i'_3}\theta_{i'_4}^2,$$

where as $W_{i_3j_3} = W_{i'_3j'_3}$, the right hand side

$$\leq C\alpha^2\theta_{i_1}\theta_{i_2}\theta_{i_3}^{1+c_1}\theta_{j_3}^{c_2}\theta_{i_4}^2\theta_{i'_4}^2,$$

with the same c_1, c_2 as in the proof of Case B. Combining these and using $v \asymp \|\theta\|_1^2$, we have that under both the null and the alternative, the contribution of Case (C) to $\mathbb{E}[X_{a1}^2]$

$$\leq C\alpha^2(\|\theta\|_1)^{-6} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i'_1,i'_2,i'_4(dist)}} \sum_{j_3(j_3\neq i_3)} \theta_{i_1}^2\theta_{i_2}^2\theta_{i_3}^{2+c_1}\theta_{j_3}^{1+c_2}\theta_{i_4}^2\theta_{i'_1}^2\theta_{i'_2}^2\theta_{i'_4}^2,$$

where the right hand size

$$\leq C\alpha^2\cdot[\|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5 + \|\theta\|^{12}\|\theta\|_3^6/\|\theta\|_1^6] \leq C\alpha^2\|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5. \tag{216}$$

Here we have again used $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$.

Combining (214), (215), and (216) gives

$$\mathbb{E}[X_{a1}^2] \leq C\alpha^2(\|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6 + \|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^5 + \|\theta\|^8\|\theta\|_3^9/\|\theta\|^5] \leq C\alpha^2\|\theta\|^8\|\theta\|_3^9/\|\theta\|_1^5,$$

where we have used $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$ and $\|\theta\| \to \infty$. This proves (210).

We now consider $\mathbb{E}[X_b^2]$. Write

$$\mathbb{E}[X_b^2] = v^{-3} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ i'_1,i'_2,i'_3,i'_4(dist)}} \sum_{\substack{j_3,j'_3\\ j_3\neq i_3, j'_3\neq i'_3}} \eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i'_2}\eta_{i'_3}\eta_{i'_4}$$
$$\mathbb{E}[W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i'_1j'_1}W_{i'_2j'_2}W_{i'_3j'_3}]\widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i'_1i'_4}.$$

Consider

$$W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}, \qquad \text{and} \qquad W_{i'_1j'_1}W_{i'_2j'_2}W_{i'_3j'_3}.$$

Each term has a mean 0, and two terms are uncorrelated with each other if and only if the two sets of random variables $\{W_{i_1j_1}, W_{i_2j_2}, W_{i_3j_3}\}$ and $\{W_{i'_1j'_1}, W_{i'_2j'_2}, W_{i'_3j'_3}\}$ are identical (however, it is possible that $W_{i_1j_1}$ does not equal to $W_{i'_1j'_1}$ but equals to $W_{i'_2j'_2}$, say). When this happens, first, $\{i_1, i_2, i_3, j_1, j_2, j_3\} = \{i'_1, i'_2, i'_3, j'_1, j'_2, j'_3\}$. Recall that $|\widetilde{\Omega}_{ij}| \leq C\alpha\theta_i\theta_j$ for all $1 \leq i, j \leq n$, $i \neq j$, and that $0 < \eta_i \leq C\theta_i$ for all $1 \leq i \leq n$. For integers $a_i \in \{0, 1\}$, $1 \leq i \leq 4$, that satisfy $\sum_{i=1}^6 a_i = 3$, we have

$$|\eta_{i_2}\eta_{i_3}\eta_{i_4}\eta_{i'_2}\eta_{i'_3}\eta_{i'_4}\widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i'_1i'_4}| \leq C\eta_{i_1}^{a_1}\eta_{j_1}^{a_2}\eta_{i_2}^{1+a_3}\eta_{j_2}^{a_4}\eta_{i_3}^{1+a_5}\eta_{j_3}^{a_6}\eta_{i_4}\eta_{i'_4}|\widetilde{\Omega}_{i_1i_4}||\widetilde{\Omega}_{i'_1i'_4}|$$
$$\leq C\alpha^2\theta_{i_1}^{1+a_1}\eta_{j_1}^{a_2}\eta_{i_2}^{1+a_3}\eta_{j_2}^{a_4}\eta_{i_3}^{1+a_5}\eta_{j_3}^{a_6}\eta_{i_4}^2\eta_{i'_4}^2.$$

Second,

$$\mathbb{E}[W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i'_1j'_1}W_{i'_2j'_2}W_{i'_3j'_3}] = \mathbb{E}[W_{i_1j_1}^2W_{i_2j_2}^2W_{i_3j_3}^2],$$

where by similar arguments, the right hand side

$$\leq C\Omega_{i_1j_1}\Omega_{i_2j_2}\Omega_{i_3j_3} \leq C\theta_{i_1}\theta_{j_1}\theta_{i_2}\theta_{j_2}\theta_{i_3}\theta_{j_3}.$$

Recall that $v \sim \|\theta\|_1^2$. Combining these gives

$$\mathbb{E}[X_b^2] \leq C\alpha^2\|\theta\|_1^{-6} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{i'_4} \sum_{\substack{j_1,j_2,j_3\\ j_1\neq i_1, j_2\neq i_2, j_3\neq i_3}} \sum_a \theta_{i_1}^{2+a_1}\eta_{j_1}^{1+a_2}\eta_{i_2}^{2+a_3}\eta_{j_2}^{1+a_4}\eta_{i_3}^{2+a_5}\eta_{j_3}^{1+a_6}\eta_{i_4}^2\eta_{i'_4}^2,$$

where $a = (a_1, a_2, \ldots, a_6)$ as above. By the way a_i are defined, the right hand side

$$\leq C\alpha^2\|\theta\|^4(\sum_a \|\theta\|_{a_1+2}^{a_1+2}\cdot\|\theta\|_{a_2+1}^{a_2+1}\cdot\|\theta\|_{a_3+2}^{a_3+2}\cdot\|\theta\|_{a_4+1}^{a_4+1}\|\theta\|_{a_5+2}^{a_5+2}\|\theta\|_{a_6+1}^{a_6+1})/\|\theta\|_1^6,$$

which by $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$, the term in the bracket does not exceed

$$C\max\{\|\theta\|^{12}, \|\theta\|_1\|\theta\|^8\|\theta\|_3^3, \|\theta\|_1^2\|\theta\|^4\|\theta\|_3^6, \|\theta\|_1^3\|\theta\|_3^9\} \leq C\|\theta\|_1^3\|\theta\|_3^9.$$

Combining these gives

$$\mathbb{E}[X_b^2] \leq C\alpha^2\|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^3. \tag{217}$$

Finally, inserting (213)-(217) into (209) gives

$$\mathrm{Var}(T_{2a}) \leq C\alpha^2[\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5 + \|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^3] \leq C\alpha^2\|\theta\|^4\|\theta\|_3^9/\|\theta\|_1^3,$$

and (138) follows.

Consider Item (g) and Item (h). The proof are similar, so we only show Item (g). The goal is to show (140). Recall that

$$T_{2c} = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\left[(\eta_{i_2} - \tilde{\eta}_{i_2})^2(\eta_{i_3} - \tilde{\eta}_{i_3})\right] \cdot \widetilde{\Omega}_{i_4 i_1}, \tag{218}$$

and

$$\widetilde{\eta} - \eta = v^{-1/2}W1_n.$$

Plugging this into T_{2c} gives (note symmetry in $\widetilde{\Omega}$)

$$\begin{aligned} T_{2c} &= -\frac{1}{v^{2/3}} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\Big(\sum_{j_2\neq i_2} W_{i_2j_2}\Big)\Big(\sum_{\ell_2\neq i_2} W_{i_2\ell_2}\Big)\Big(\sum_{j_3\neq i_3} W_{i_3j_3}\Big)\widetilde{\Omega}_{i_4i_1} \\ &= -\frac{1}{v^{3/2}} \sum_{\substack{i_1,i_2,i_3,i_4(dist)\\ j_1\neq i_1, j_2\neq i_2, j_3\neq i_3}} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}\widetilde{\Omega}_{i_1i_4}. \end{aligned}$$

By basic combinatorics and careful observations, we have

$$W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3} = \begin{cases} W_{i_2i_3}^3, & \text{if } j_1 = \ell_2 = i_3, j_3 = i_2, \\ W_{i_2j_2}^2 W_{i_3j_3}, & \text{if } j_1 = \ell_2, (j_2, j_3) \neq (i_3, i_2), \\ W_{i_2i_3}^2 W_{i_2\ell_2}, & \text{if } j_2 = i_3, j_3 = i_2, \ell_2 \neq i_3, \\ W_{i_2i_3}^2 W_{i_2j_2}, & \text{if } \ell_2 = i_3, j_3 = i_2, j_2 \neq i_3, \\ W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}, & \text{otherwise.} \end{cases} \tag{219}$$

This allows us to further split T_{2c} into 4 different terms:

$$T_{2c} = Y_a + Y_{b1} + Y_{b2} + Y_{b3} + Y_c, \tag{220}$$

$$Y_a = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_3\neq i_3} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2i_3}^3\widetilde{\Omega}_{i_1i_4},$$

$$Y_{b1} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_3\neq i_3} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2j_2}^2W_{i_3j_3}\widetilde{\Omega}_{i_1i_4},$$

$$Y_{b2} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_2\neq i_2} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2i_3}^2W_{i_2\ell_2}\widetilde{\Omega}_{i_1i_4},$$

$$Y_{b3} = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_1\neq i_1} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2i_3}^2W_{i_2j_2}\widetilde{\Omega}_{i_1i_4},$$

$$Y_c = -\frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_2,\ell_2,j_3\\ j_2\neq i_2,\ell_2\neq i_2,j_3\neq i_3\\ j_2\neq i_3,\ell_2\neq i_3,j_3\neq i_2}} \eta_{i_1}\eta_{i_3}\eta_{i_4}W_{i_2j_2}W_{i_2\ell_2}W_{i_3j_3}\widetilde{\Omega}_{i_1i_4}.$$

We now show the two claims in (140) separately. Consider the first claim. It is seen that out of the 5 terms on the right hand side of (220), the mean of all terms are 0, except for the first one. Note that for any $1 \le i, j \le n, i \ne j$, $\mathbb{E}[W_{ij}^3] \le C\Omega_{ij}$. Together with $\Omega_{ij} \le C\theta_i\theta_j$, $\widetilde{\Omega}_{ij} \le C\alpha\theta_i\theta_j$, $0 < \eta_i < C\theta_i$ and $v \sim \|\theta\|_1^2$, it follows

$$\begin{aligned}\mathbb{E}[|Y_a|] &\le \frac{1}{v^{3/2}} \sum_{i_1,i_2,i_3,i_4 (dist)} \eta_{i_1}\eta_{i_3}\eta_{i_4}\Omega_{i_2i_3}\widetilde{\Omega}_{i_1i_4} \\ &\le C\alpha \cdot \frac{1}{\|\theta\|_1^3} \sum_{i_1,i_2,i_3,i_4 (dist)} \theta_{i_1}^2\theta_{i_2}\theta_{i_3}^2\eta_{i_4}^2,\end{aligned}$$

where the last term is no greater than $C\alpha \cdot \|\theta\|^6/\|\theta\|_1^3$, and the first claim of (140) follows.

Consider the second claim of (140). By Cauchy-Schwartz inequality,

$$\begin{aligned}\mathrm{Var}(T_{2c}) &\le C(\mathrm{Var}(Y_a) + \mathrm{Var}(Y_{b1}) + \mathrm{Var}(Y_{b2}) + \mathrm{Var}(Y_{b3}) + \mathrm{Var}(Y_c)) \\ &\le C(\mathrm{Var}(Y_a) + \mathbb{E}[Y_{b1}^2] + \mathbb{E}[Y_{b2}^2] + \mathbb{E}[Y_{b3}^2] + \mathbb{E}[Y_c^2]).\end{aligned} \tag{221}$$

We now study $\mathrm{Var}(Y_a)$. Write

$$\mathrm{Var}(Y_a) = v^{-3} \sum_{\substack{i_1,i_2,i_3,i_4 (dist) \\ i_1',i_2',i_3',i_4' (dist)}} \eta_{i_1}\eta_{i_3}\eta_{i_4}\eta_{i_1'}\eta_{i_3'}\eta_{i_4'}\mathbb{E}[(W_{i_2i_3}^3 - \mathbb{E}[W_{i_2i_3}^3])(W_{i_2'i_3'}^3 - \mathbb{E}[W_{i_2'i_3'}^3])] \cdot \widetilde{\Omega}_{i_1i_4}\widetilde{\Omega}_{i_1'i_4'}.$$

Fix a term $(W_{i_2i_3}^3 - \mathbb{E}[W_{i_2i_3}^3])(W_{i_2'i_3'}^3 - \mathbb{E}[W_{i_2'i_3'}^3])$. When the mean is nonzero, we must have $\{i_2, i_3\} = \{i_2', i_3'\}$, and when this happens,

$$\mathbb{E}[(W_{i_2i_3}^3 - \mathbb{E}[W_{i_2i_3}^3])(W_{i_2'i_3'}^3 - \mathbb{E}[W_{i_2'i_3'}^3])] = \mathrm{Var}(W_{i_2i_3}^3).$$

For a random variable X, we have $\mathrm{Var}(X) \le \mathbb{E}[X^2]$, and it follows that

$$\mathrm{Var}(W_{i_2i_3}^3) \le \mathbb{E}[W_{i_2i_3}^6] \le \mathbb{E}[W_{i_2i_3}^2],$$

where we have used the property that $0 \le W_{i_2i_3}^2 \le 1$. Notice that $\mathbb{E}[W_{i_2i_3}^2] \le C\theta_{i_2}\theta_{i_3}$, and recall that $v \asymp \|\theta\|_1^2$, $\widetilde{\Omega}_{ij} \le C\alpha\theta_i\theta_j$ and $0 < \eta_i \le C\theta_i$ for all $1 \le i \le n$. Combining these gives

$$\mathrm{Var}(Y_a) \le C\alpha^2(\|\theta\|_1^{-6}) \cdot \sum_{\substack{i_1,i_2,i_3,i_4 (dist) \\ i_1',i_4' (dist)}} \theta_{i_1}^2\theta_{i_2}\theta_{i_3}^3\theta_{i_4}^2\theta_{i_1'}^2\theta_{i_4'}^2 \le C\alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5. \tag{222}$$

Additionally, note that

- The proof of Y_{b1}, Y_{b2}, and Y_{b3} is similar to that of X_{a1} in Item (e).
- The proof of Y_c is similar to that of X_b in Item (e).

For these reasons, we skip the proof details, but only to state that, both under the null and the alternative,

$$\mathbb{E}[Y_{b1}^2] \le C\alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1, \tag{223}$$

$$\mathbb{E}[Y_{b2}^2] \le C\alpha^2\|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5, \tag{224}$$

$$\mathbb{E}[Y_{b3}^2] \le C\alpha^2\|\theta\|^{12}\|\theta\|_3^3/\|\theta\|_1^5, \tag{225}$$

and therefore,

$$\mathbb{E}[Y_{b1}^2] + \mathbb{E}[Y_{b2}^2] + \mathbb{E}[Y_{b3}^2] \le C\alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1. \tag{226}$$

At the same time, both under the null and the alternative,

$$\mathbb{E}[Y_c^2] \le C\alpha^2 \cdot \|\theta\|^{10}\|\theta\|_3^3/\|\theta\|_1^3. \tag{227}$$

Inserting (226) and (227) into (221) gives

$$\mathbb{E}[T_{2c}^2] \le C\alpha^2[\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5 + \|\theta\|^8\|\theta\|_3^3/\|\theta\|_1 + \|\theta\|^{10}\|\theta\|_3^3/\|\theta\|_1^3] \le C\alpha^2\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1.$$

This proves (140).

Consider Item (i). The goal is to show (142). Recall that

$$F_a = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1}-\tilde{\eta}_{i_1})(\eta_{i_2}-\tilde{\eta}_{i_2})(\eta_{i_3}-\tilde{\eta}_{i_3})(\eta_{i_4}-\tilde{\eta}_{i_4})\big], \tag{228}$$

and that for any $1 \le i \le n$,

$$\tilde{\eta}_i - \eta_i = v^{-1/2}\sum_{j\neq i}^{n} W_{ij}.$$

Inserting it into (228) gives

$$F_a = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}\big[(\eta_{i_1}-\tilde{\eta}_{i_1})(\eta_{i_2}-\tilde{\eta}_{i_2})(\eta_{i_3}-\tilde{\eta}_{i_3})(\eta_{i_4}-\tilde{\eta}_{i_4})\big],$$

By basic combinatorics and basic algebra, we have

$$W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i_4j_4} = \begin{cases} W_{i_1i_2}^2W_{i_3i_4}^2, & \text{if } (i_1,j_1)=(j_2,i_2),(i_3,j_3)=(j_4,i_4),\\ W_{i_1i_3}^2W_{i_2i_4}^2, & \text{if } (i_1,j_1)=(j_3,i_3),(i_2,j_2)=(j_4,i_4),\\ W_{i_1i_4}^2W_{i_2i_3}^2, & \text{if } (i_1,i_4)=(j_4,i_1),(i_2,j_2)=(j_3,i_3),\\ W_{i_1i_2}^2W_{i_3j_3}W_{i_4j_4}, & \text{if } (i_1,j_1)=(j_2,i_2),(j_4,j_3)\neq(i_3,i_4),\\ W_{i_1i_3}^2W_{i_2j_2}W_{i_4j_4}, & \text{if } (i_1,j_1)=(j_3,i_3),(j_4,j_2)\neq(i_2,i_4),\\ W_{i_1i_4}^2W_{i_2j_2}W_{i_3j_4}, & \text{if } (i_1,j_1)=(j_4,i_4),(j_3,j_2)\neq(i_2,i_3),\\ W_{i_2i_3}^2W_{i_1j_1}W_{i_4j_4}, & \text{if } (i_2,j_2)=(j_3,i_3),(j_4,j_1)\neq(i_1,i_4),\\ W_{i_2i_4}^2W_{i_1j_1}W_{i_3j_3}, & \text{if } (i_2,j_2)=(j_4,i_4),(j_3,j_1)\neq(i_1,i_3),\\ W_{i_3i_4}^2W_{i_1j_1}W_{i_2j_2}, & \text{if } (i_3,j_3)=(j_4,i_4),(j_2,j_1)\neq(i_1,i_2).\\ W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i_4j_4}, & \text{otherwise.} \end{cases}$$

By symmetry, it allows us to further split F_1 into 3 different terms:

$$F_1 = 3X_a + 6X_b + X_c, \tag{229}$$

where

$$X_a = v^{-2}\sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}W_{i_1i_2}^2W_{i_3i_4}^2,$$

$$X_b = v^{-2}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{\substack{j_3,j_4\\(j_3,j_4)\neq(i_4,i_3)}} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}W_{i_1i_2}^2W_{i_3j_3}W_{i_4j_4},$$

and

$$X_c = v^{-2}\sum_{i_1,i_2,i_3,i_4(dist)}\sum_{\substack{j_1,j_2,j_3,j_4\\ j_k\neq i_\ell, k,\ell=1,2,3,4}} \eta_{i_1}\eta_{i_2}\eta_{i_3}\eta_{i_4}W_{i_1j_1}W_{i_2j_2}W_{i_3j_3}W_{i_4j_4}.$$

We now show the two claims in (142) separately. Consider the first claim of (142). Note that $\mathbb{E}[X_b] = \mathbb{E}[X_c] = 0$. Recall that both under the null and the alternative, for any $i \neq j$, $\mathbb{E}[W_{ij}^2] = \Omega_{ij}(1 - \Omega_{ij}) \leq C\theta_i\theta_j$, and that $0 < \eta_i \leq C\theta_i$, and that $v \asymp \|\theta\|_1^2$, Therefore,

$$0 < \mathbb{E}[X_a] \leq v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4} \leq C\|\theta\|^8/\|\theta\|_1^4.$$

Inserting into (229) gives

$$\mathbb{E}[|F_1|] \leq C\|\theta\|^8/\|\theta\|_1^4,$$

and the first claim (142) follows.

Consider the second claim (142) next. By (229) and Cauchy-Schwarz inequality,

$$\text{Var}(F_1) \leq C(\text{Var}(X_a) + \text{Var}(X_b) + \text{Var}(X_c)) \leq C(\text{Var}(X_a) + \mathbb{E}[X_b^2] + \mathbb{E}[X_c^2]). \tag{230}$$

We now consider $\text{Var}(X_a)$, $\mathbb{E}[X_b^2]$, and $\mathbb{E}[X_c^2]$, separately. Note that

- The proof of $\text{Var}(X_a)$ is similar to that of $\text{Var}(X_a)$ in Item (a).
- The proof of $\mathbb{E}[X_b^2]$ is similar to that of $\sum_{s=1}^4 \mathbb{E}[X_{bs}^2]$ in Item (a).
- The proof of $\mathbb{E}[X_c^2]$ is similar to that of $\mathbb{E}[X_c^2]$ in Item (a).

For these reasons, we omit the proof details and only state the claims. We have that under both the null and the alternative,

$$\text{Var}(X_a) \leq C\|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^8. \tag{231}$$

$$\text{Var}(X_b^2) + \text{Var}(Y_{a3}) \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \tag{232}$$

$$\mathbb{E}[X_c^2] \leq C\|\theta\|_3^{12}/\|\theta\|_1^4, \tag{233}$$

Finally, inserting (231), (232), and (233) into (229) gives that, both under the null and the alternative,

$$\text{Var}(F_1) \leq C[\|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^8 + \|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6 + \|\theta\|_3^{12}/\|\theta\|_1^4] \leq C\|\theta\|^8\|\theta\|_3^6/\|\theta\|_1^6,$$

where we have used $\|\theta\| \to \infty$ and $\|\theta\|_3^3 \ll \|\theta\|^2 \ll \|\theta\|_1$. This gives (142) and completes the proof for Item (i).

Consider Item (j). The goal is to show (143). Recall that

$$F_b = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta_{i_3}^2\eta_{i_4}\left[(\eta_{i_1} - \tilde{\eta}_{i_1})^2(\eta_{i_2} - \tilde{\eta}_{i_2})(\eta_{i_4} - \tilde{\eta}_{i_4})\right],$$

and that

$$\widetilde{\eta} - \eta = v^{-1/2}W1_n.$$

Plugging this into F_b, we have

$$F_b = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,\ell_1,j_2,j_4 \\ j_1\neq i_1,\ell_1\neq i_1,j_2\neq i_2,j_4\neq i_4}} \eta_{i_2}\eta_{i_3}^2\eta_{i_4}W_{i_1j_1}W_{i_1\ell_1}W_{i_2j_2}W_{i_4j_4}.$$

By basic combinatorics and basic algebra, we have

$$W_{i_1j_1}W_{i_1\ell_1}W_{i_2j_2}W_{i_4j_4} = \begin{cases} W^3_{i_1i_2}W_{i_4j_4}, & \text{if } j_1,\ell_1 = i_2, j_2 = i_1, \\ W^3_{i_1i_4}W_{i_2j_2}, & \text{if } j_1,\ell_1 = i_4, j_4 = i_1, \\ W^2_{i_1i_2}W^2_{i_1i_4}, & \text{if } (j_1,j_2) = (i_2,i_1), (\ell_1,j_4) = (i_4,i_1), \\ W^2_{i_1i_2}W^2_{i_1i_4}, & \text{if } (\ell_1,j_2) = (i_2,i_1), (j_1,j_4) = (i_4,i_1), \\ W^2_{i_1i_4}W^2_{i_1i_2}, & \text{if } (j_1,j_4) = (i_4,i_1), (\ell_1,j_2) = (i_2,i_1), \\ W^2_{i_1i_4}W^2_{i_1i_2}, & \text{if } (\ell_1,j_4) = (i_4,i_1), (j_1,j_2) = (i_2,i_1), \\ W^2_{i_1j_1}W^2_{i_2i_4}, & \text{if } j_1 = \ell_1, (j_2,j_4) = (i_4,i_2), \\ W^2_{i_1i_2}W_{i_1j_1}W_{i_4j_4}, & \text{if } \ell_1 = i_2, j_2 = i_1, j_1 \neq i_2, i_4, \\ W^2_{i_1i_2}W_{i_1\ell_1}W_{i_4j_4}, & \text{if } j_1 = i_2, j_2 = i_1, \ell_1 \neq i_2, i_4, \\ W^2_{i_1i_4}W_{i_1j_1}W_{i_2j_2}, & \text{if } \ell_1 = i_4, j_4 = i_1, \ell_1 \neq i_2, i_4, \\ W^2_{i_1i_4}W_{i_1\ell_1}W_{i_2j_2}, & \text{if } j_1 = i_4, j_4 = i_1, j_1 \neq i_2, i_4, \\ W^2_{i_2i_4}W_{i_1j_1}W_{i_1\ell_1}, & \text{if } j_1 \neq \ell_1, (j_2,j_4) = (i_4,i_2). \\ W^2_{i_1j_1}W_{i_2j_2}W_{i_4j_4}, & \text{if } j_1 = \ell_1, (j_1,j_2) \neq (i_2,i_1), (j_1,j_4) \neq (i_4,i_1), \\ W_{i_1j_1}W_{i_1\ell_1}W_{i_2j_2}W_{i_4j_4}, & \text{otherwise.} \end{cases}$$

By these and symmetry, we can further split F_b into 7 different terms,

We decompose

$$F_b = 2Y_{a1} + 4Y_{a2} + Y_{a3} + 4Y_{b1} + Y_{b2} + Y_{b3} + Y_c, \tag{234}$$

where

$$Y_{a1} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_4, j_4\neq i_4} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^3_{i_1i_2}W_{i_4j_4},$$

$$Y_{a2} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^2_{i_1i_2}W^2_{i_1i_4},$$

$$Y_{a3} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_1, j_1\neq i_1} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^2_{i_1j_1}W^2_{i_2i_4},$$

$$Y_{b1} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,j_4 \\ j_1\neq i_1, j_4\neq i_4}} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^2_{i_1i_2}W_{i_1j_1}W_{i_4j_4},$$

$$Y_{b2} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,\ell_1 \\ j_1,\ell_1\neq i_1}} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^2_{i_2i_4}W_{i_1j_1}W_{i_1\ell_1},$$

$$Y_{b3} = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,j_2,j_4 \\ j_1\neq i_1, j_2\neq i_2, j_4\neq i_4}} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W^2_{i_1j_1}W_{i_2j_2}W_{i_4j_4},$$

$$Y_c = v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{\substack{j_1,\ell_1,j_2,j_4 \\ j_1,\ell_1\notin\{i_1,i_2,i_4\} \\ j_2\notin\{i_1,i_4\}, j_4\notin\{i_1,i_2\}}} \eta_{i_2}\eta^2_{i_3}\eta_{i_4}W_{i_1j_1}W_{i_1\ell_1}W_{i_2j_2}W_{i_4j_4},$$

We now consider the two claims in (143) separately. Consider the first claim. It is seen that only the second and the third terms above have non-zero mean. Recall that both under the null and the alternative, for any $i \neq j$, $\mathbb{E}[W_{ij}^2] = \Omega_{ij}(1-\Omega_{ij}) \leq C\theta_i\theta_j$, $0 < \eta_i \leq C\theta_i$, and that $v \asymp \|\theta\|_1^2$. It follows

$$0 < \mathbb{E}[Y_{a2}] \leq v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \theta_{i_2}\theta_{i_3}^2\theta_{i_4} \cdot \theta_{i_1}^2\theta_{i_2}\theta_{i_4} \leq C\|\theta\|^8/\|\theta\|_1^4. \tag{235}$$

and

$$0 < \mathbb{E}[Y_{a3}] \leq v^{-2} \sum_{i_1,i_2,i_3,i_4(dist)} \sum_{j_1} \theta_{i_2}\theta_{i_3}^2\theta_{i_4} \cdot \theta_{i_1}\theta_{i_2}\theta_{j_1}\theta_{i_4} \leq C\|\theta\|^6/\|\theta\|_1^2. \tag{236}$$

Combining (235), (236) with (234) gives

$$\mathbb{E}[|F_2|] \leq C[\|\theta\|^8/\|\theta\|_1^4 + \|\theta\|^6/\|\theta\|_1^2] \leq C\|\theta\|^6/\|\theta\|_1^2,$$

where we've used the universal inequality that $\|\theta\|^2 \leq \|\theta\|_1$. It follows the first claim of (143).

We now show the second claim of (143). By Cauchy-Schwarz inequality,

$$\begin{aligned} \mathrm{Var}(F_b) &\leq C\big(\mathrm{Var}(Y_{a1}) + \mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3}) + \mathrm{Var}(Y_{b1}) + \mathrm{Var}(Y_{b2}) + \mathrm{Var}(Y_{b3}) + \mathrm{Var}(Y_c)\big) \\ &\leq C\big(\mathrm{Var}(Y_{a1}) + \mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3}) + \mathbb{E}[Y_{b1}^2] + \mathbb{E}[Y_{b2}^2] + \mathbb{E}[Y_{b3}^2] + \mathbb{E}[Y_c^2]\big). \end{aligned} \tag{237}$$

We now consider $\mathrm{Var}(Y_{a1})$, $\mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3})$, $\mathbb{E}[Y_{b1}^2] + \mathbb{E}[Y_{b2}^2] + \mathbb{E}[Y_{b3}^2]$, and $\mathbb{E}[Y_c^2]$, separately. Note that

- The proof of $\mathrm{Var}(Y_{a1})$ is similar to that of $\mathrm{Var}(Y_a)$ in Item (b).
- The proof of $\mathrm{Var}(Y_{a2})$ and $\mathrm{Var}(Y_{a3})$ are similar to that of $\mathrm{Var}(X_a)$ in Item (a).
- The proof of $\sum_{s=1}^3 \mathbb{E}[Y_{b_s}^2]$ is similar to that of $\sum_{s=1}^4 \mathbb{E}[X_{bs}^2]$ in Item (a).
- The proof of $\mathbb{E}[Y_c^2]$ is similar to that of $\mathbb{E}[X_c^2]$ in Item (a).

For these reasons, we omit the proof details and only state the claims. We have that under both the null and the alternative,

$$\mathrm{Var}(Y_{a1}) \leq C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5. \tag{238}$$

$$\mathrm{Var}(Y_{a2}) + \mathrm{Var}(Y_{a3}) \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \tag{239}$$

$$\sum_{s=1}^{3} \mathbb{E}[Y_{b_s}^2] \leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2, \tag{240}$$

$$\mathbb{E}[Y_c^2] \leq C\|\theta\|^6\|\theta\|_3^6/\|\theta\|_1^4. \tag{241}$$

Finally, inserting (238), (239), (240), and (241) into (237) gives

$$\begin{aligned} \mathrm{Var}(F_2) &\leq C[\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4 + \|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^2 + \|\theta\|^6\|\theta\|_3^6/\|\theta\|_1^4] \\ &\leq C\|\theta\|^4\|\theta\|_3^6/\|\theta\|_1^4, \end{aligned} \tag{242}$$

where we have used $\|\theta\|_3^3 \ll \|\theta\|^2 \ll \|\theta\|_1$, $\|\theta\| \to \infty$ and $\|\theta\|^4 \leq \|\theta\|_1\|\theta\|_3^3$. This completes the proof of (143).

Consider Item (k). The goal is to show (144) and (145). Recall that

$$F_c = \sum_{i_1,i_2,i_3,i_4(dist)} \eta_{i_2}^2\eta_{i_4}^2\big[(\eta_{i_1} - \tilde{\eta}_{i_1})^2(\eta_{i_3} - \tilde{\eta}_{i_3})^2\big],$$

and that $\widetilde{\eta} - \eta = v^{-1/2} W 1_n$. Plugging this into F_3 gives

$$F_c = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_1,\ell_1,j_2,j_4 \\ j_1 \neq i_1, \ell_1 \neq i_1, j_3 \neq i_3, \ell_3 \neq i_3}} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 j_1} W_{i_1 \ell_1} W_{i_3 j_3} W_{i_3 \ell_3}.$$

By basic combinatorics and basic algebra, we have

$$W_{i_1 j_1} W_{i_1 \ell_1} W_{i_3 j_3} W_{i_3 \ell_3} = \begin{cases} W_{i_1 i_3}^4, & \text{if } j_1 = \ell_1 = i_1, j_3 = \ell_3 = i_1, \\ W_{i_1 i_3}^3 W_{i_1 j_1}, & \text{if } j_3 = \ell_3 = i_1, \ell_1 = i_3, \\ W_{i_1 i_3}^3 W_{i_1 \ell_1}, & \text{if } j_3 = \ell_3 = i_1, j_1 = i_3, \\ W_{i_1 i_3}^3 W_{i_3 j_3}, & \text{if } j_1 = \ell_1 = i_3, \ell_3 = i_1, \\ W_{i_1 i_3}^3 W_{i_3 \ell_3}, & \text{if } j_1 = \ell_1 = i_3, j_3 = i_1, \\ W_{i_1 j_1}^2 W_{i_3 j_3}^2, & \text{if } j_1 = \ell_1, j_3 = \ell_3, \\ W_{i_1 j_1}^2 W_{i_3 j_3} W_{i_3 \ell_3}, & \text{if } j_1 = \ell_1 \neq i_3, j_3 \neq \ell_3, \\ W_{i_3 j_3}^2 W_{i_1 j_1} W_{i_1 \ell_1}, & \text{if } j_3 = \ell_3 \neq i_1, j_1 \neq \ell_1, \\ W_{i_1 i_3}^2 W_{i_1 \ell_1} W_{i_3 \ell_3}, & \text{if } j_1 = i_3, j_3 = i_1, \\ W_{i_1 i_3}^2 W_{i_1 j_1} W_{i_3 j_3}, & \text{if } \ell_1 = i_3, \ell_3 = i_1, \\ W_{i_1 i_3}^2 W_{i_1 j_1} W_{i_3 \ell_3}, & \text{if } \ell_1 = i_3, j_3 = i_1, \\ W_{i_1 i_3}^2 W_{i_1 \ell_1} W_{i_3 j_3}, & \text{if } j_1 = i_3, \ell_3 = i_1, \\ W_{i_1 j_1} W_{i_1 \ell_1} W_{i_3 j_3} W_{i_3 \ell_3}, & \text{otherwise.} \end{cases}$$

By these and symmetry, we can further split F_3 into 6 different terms:

$$F_c = Z_a + 4Z_{b1} + Z_{b2} + 2Z_{c1} + 4Z_{c2} + Z_d, \tag{243}$$

where

$$Z_a = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 i_3}^4,$$

$$Z_{b1} = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{j_4, j_4 \neq i_4} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 i_3}^3 W_{i_3 j_3},$$

$$Z_{b2} = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{j_1, j_1 \neq i_1, j_3, j_3 \neq i_3} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 j_1}^2 W_{i_3 j_3}^2,$$

$$Z_{c1} = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_1, j_3, \ell_3 \\ j_1 \notin \{i_1, i_3\}, j_3, \ell_3}} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 j_1}^2 W_{i_3 j_3} W_{i_3 \ell_3},$$

$$Z_{c2} = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{\ell_1, \ell_3 \\ \ell_1 \neq i_1, \ell_3 \neq i_3}} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 i_3}^2 W_{i_1 \ell_1} W_{i_3 \ell_3},$$

$$Z_d = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{\substack{j_1, \ell_1, j_3, \ell_3 \\ j_1 \neq \ell_1, j_3 \neq \ell_3 \\ j_1, \ell_1 \neq i_3, j_3, \ell_3 \neq i_1}} \eta_{i_2}^2 \eta_{i_4}^2 W_{i_1 j_1} W_{i_1 \ell_1} W_{i_3 j_3} W_{i_3 \ell_3}.$$

We now show (144) and (145) separately. Consider (144) first. It is among all the 6 Z-terms, only Z_a and Z_{b2} have non-zero means. We now consider $\mathbb{E}[Z_a]$ and $\mathbb{E}[Z_{b2}]$ separately.

First, consider $\mathbb{E}[Z_a]$. By similar arguments, both under the null and the alternative,

$$\mathbb{E}[W_{i_1 i_3}^4] \leq C\Omega_{i_1 i_3} \leq C\theta_{i_1}\theta_{i_3}.$$

Recalling that $0 < \eta_i \leq C\theta_i$ and $v \asymp \|\theta\|^2$, it is seen that

$$\mathbb{E}[Z_a] \leq C(\|\theta\|_1)^{-4} \sum_{i_1,i_2,i_3,i_4 (dist)} \theta_{i_2}^2 \theta_{i_4}^2 \theta_{i_1} \theta_{i_3} \leq C\|\theta\|^4/\|\theta\|_1^2. \tag{244}$$

Next, consider $\mathbb{E}[Z_{b2}]$. First, recall that under the null, $\Omega = \theta\theta'$, $v = 1_n'(\Omega - \text{diag}(\Omega))1_n$, and $\eta = v^{-1/2}(\Omega - \text{diag}(\Omega)1_n$. It is seen $v \sim \|\theta\|_1^2$, $\eta_i = (1 + o(1)\theta_i$, $1 \leq i \leq n$, where $o(1) \to 0$ uniformly for all $1 \leq i \leq n$, and for any $i \neq j$, $\mathbb{E}[W_{ij}^2] = (1 + o(1))\theta_i\theta_j$, where $o(1) \to 0$ uniformly for all $1 \leq i, j \leq n$. It follows

$$\mathbb{E}[Z_{b2}] = v^{-2} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{j_1, j_1 \neq i_1, j_3, j_3 \neq i_3} \eta_{i_2}^2 \eta_{i_4}^2 \mathbb{E}[W_{i_1 j_1}^2 W_{i_3 j_3}^2], \tag{245}$$

which

$$\sim (\|\theta\|_1)^{-4} \sum_{i_1,i_2,i_3,i_4 (dist)} \sum_{j_1, j_1 \neq i_1, j_3, j_3 \neq i_3} \theta_{i_1} \theta_{i_2}^2 \theta_{i_3} \theta_{i_4}^2 \theta_{j_1} \theta_{j_3} \sim \|\theta\|^4.$$

Second, under the alternative, by similar argument, we have that $v \asymp \|\theta\|_1^2$, $0 < \eta_i < C\theta_i$ for all $1 \leq i \leq n$, and $\mathbb{E}[W_{ij}^2] \leq C\theta_i\theta_j$ for all $1 \leq i, j \leq n$, $i \neq j$. Similar to that under the null, we have

$$0 < |\mathbb{E}[Z_{b2}]| \leq C\|\theta\|^4. \tag{246}$$

Inserting (244), (245), and (246) into (243) and recalling that the mean of all other Z terms are 0,

$$\mathbb{E}[F_3] \sim \|\theta\|^4, \qquad \text{under the null,}$$

and

$$\mathbb{E}[F_3] \leq C\|\theta\|^4, \qquad \text{under the alternative,}$$

where we have used $\|\theta\|_1 \to \infty$. This proves (144).

We now consider (145). By Cauchy-Schwarz inequality,

$$\begin{aligned} \text{Var}(F_c) &\leq C\big(\text{Var}(Z_a) + \text{Var}(Z_{b1}) + \text{Var}(Z_{b2}) + \text{Var}(Z_{c1}) + \text{Var}(Z_{c2}) + \text{Var}(Z_d)\big) \\ &\leq C\big(\text{Var}(Z_a) + \mathbb{E}[Z_{b1}^2] + \text{Var}(Z_{b2}) + \mathbb{E}[Z_{c1}^2] + \mathbb{E}[Z_{c2}^2] + \mathbb{E}[Z_d^2]\big). \end{aligned} \tag{247}$$

Consider $\text{Var}(Z_a)$. Write

$$\text{Var}(Z_a) = v^{-4} \sum_{\substack{i_1,i_2,i_3,i_4 (dist) \\ i_1',i_2',i_3',i_4' (dist)}} \eta_{i_2}^2 \eta_{i_4}^2 \eta_{i_2'}^2 \eta_{i_4'}^2 \mathbb{E}[(W_{i_1 i_3}^4 - \mathbb{E}[W_{i_1 i_3}^4])(W_{i_1' i_3'}^4 - \mathbb{E}[W_{i_1' i_3'}^4])].$$

Fix a term $(W_{i_1 i_3}^4 - \mathbb{E}[W_{i_1 i_3}^4])(W_{i_1' i_3'}^4 - \mathbb{E}[W_{i_1' i_3'}^4])$. When the mean is nonzero, we must have $\{i_1, i_3\} = \{i_1', i_3'\}$, and when this happens,

$$\mathbb{E}[(W_{i_1 i_3}^4 - \mathbb{E}[W_{i_1 i_3}^4])(W_{i_1' i_3'}^4 - \mathbb{E}[W_{i_1' i_3'}^4])] = \text{Var}(W_{i_1 i_3}^4).$$

For a random variable X, we have $\text{Var}(X) \leq \mathbb{E}[X^2]$, and it follows that

$$\text{Var}(W_{i_1 i_3}^4) \leq \mathbb{E}[W_{i_1 i_3}^8] \leq \mathbb{E}[W_{i_1 i_3}^2],$$

where we have used the property that $0 \leq W_{i_1 i_3}^2 \leq 1$; note that $\mathbb{E}[W_{i_1 i_3}^2] \leq C\theta_{i_1}\theta_{i_3}$. Recall that $v \asymp \|\theta\|_1^2$ and $0 < \eta_i \leq C\theta_i$ for all $1 \leq i \leq n$. Combining these gives

$$\text{Var}(Z_a) \leq C(\|\theta\|_1^{-8}) \cdot \sum_{\substack{i_1,i_2,i_3,i_4 (dist) \\ i_2',i_4' (dist)}} \theta_{i_2}^2 \theta_{i_4}^2 \theta_{i_2'}^2 \theta_{i_4'}^2 \theta_{i_1} \theta_{i_3} \leq C\|\theta\|^8/\|\theta\|_1^6. \tag{248}$$

We now consider all other terms on the right hand side of (247). Note that

- The proof of $\mathbb{E}[Z_{b1}^2]$ is similar to that of Y_{a1} in Item (b).
- The proof of $\mathrm{Var}(Z_{b2})$ is similar to that of X_a in Item (a).
- The proof of $\mathbb{E}[Z_{c1}^2]$ and $\mathbb{E}[Z_{c2}^2]$ are similar to that of X_b in Item (a).
- The proof of $\mathbb{E}[Z_d^2]$ is similar to that of X_c in Item (a).

For these reasons, we skip the proof details. We have that, under both the null and the alternative,

$$\mathbb{E}[Z_{b1}^2] \leq C\|\theta\|^8\|\theta\|_3^3/\|\theta\|_1^5, \tag{249}$$

$$\mathrm{Var}(Z_{b2}) \leq C\|\theta\|^8/\|\theta\|_1^2, \tag{250}$$

$$\mathbb{E}[Z_{c1}^2] + \mathbb{E}[Z_{c2}^2] \leq C\|\theta\|^{10}/\|\theta\|_1^2, \tag{251}$$

and

$$\mathbb{E}[Z_d^2] \leq C\|\theta\|^{12}/\|\theta\|_1^4. \tag{252}$$

Inserting (248), (249), (250), (251) and (252) into (247) gives

$$\begin{aligned}\mathrm{Var}(F_c) &\leq C[\|\theta\|^8/\|\theta\|_1^6 + \|\theta\|^8/\|\theta\|_1^2 + \|\theta\|^{10}/\|\theta\|_1^2 + \|\theta\|^{12}/\|\theta\|_1^4] \\ &\leq C\|\theta\|^{10}/\|\theta\|_1^2,\end{aligned}$$

which completes the proof of (145).

G.4.9. *Proof of Lemma G.10.* Define an event D as

$$D = \big\{|V - v| \leq \|\theta\|_1 \cdot x_n\big\}, \qquad \text{for} \quad \sqrt{\log(\|\theta\|_1)} \ll x_n \ll \|\theta\|_1.$$

We aim to show that

$$\mathbb{E}[(Q_n - Q_n^*)^2 \cdot I_{D^c}] = o(\|\theta\|^8). \tag{253}$$

First, we bound the tail probability of $|V - v|$. Write

$$V - v = 2\sum_{i<j}(A_{ij} - \Omega_{ij}).$$

The variables $\{A_{ij} - \Omega_{ij}\}_{1\leq i<j\leq n}$ are mutually independent with mean zero. They satisfy $|A_{ij} - \Omega_{ij}| \leq 1$ and $\sum_{i<j}\mathrm{Var}(A_{ij} - \Omega_{ij}) \leq \sum_{i<j}\Omega_{ij} \leq 1_n'\Omega 1_n/2 \leq \|\theta\|_1^2/2$. Applying the Bernstein's inequality, for any $t > 0$,

$$\mathbb{P}\Big(\Big|2\sum_{i<j}(A_{ij} - \Omega_{ij})\Big| > t\Big) \leq 2\exp\Big(-\frac{t^2/2}{2\|\theta\|_1^2 + t/3}\Big).$$

We immediately have that, for some positive constants $C_1, C_2 > 0$,

$$\mathbb{P}(|V - v| > t) \leq \begin{cases} 2\exp\big(-\frac{C_1}{\|\theta\|_1^2}t^2\big), & \text{when } x_n\|\theta\|_1 \leq t \leq \|\theta\|_1^2, \\ 2\exp\big(-C_2 t\big), & \text{when } t > \|\theta\|_1^2. \end{cases} \tag{254}$$

Especially, letting $t = x_n\|\theta\|_1$, we have

$$\mathbb{P}(D^c) \leq 2\exp(-C_1 x_n^2). \tag{255}$$

Next, we derive an upper bound of $(Q_n - Q_n^*)^2$ in terms of V. Recall that V is the total number of edges and that $Q_n = \sum_{i,j,k,\ell(dist)} M_{ij}M_{jk}M_{k\ell}M_{\ell i}$, where $M_{ij} = A_{ij} - \hat{\eta}_i\hat{\eta}_j$. If one node of i, j, k, ℓ has a zero degree (say, node i), then $A_{ij} = 0$ and $\hat{\eta}_i = 0$, and it follows that $M_{ij}=0$ and $M_{ij}M_{jk}M_{k\ell}M_{\ell i}=0$. Hence, only when (i,j,k,ℓ) all have nonzero degrees, this quadruple has a contribution to Q_n. Since V is the total number of edges, there are at most V nodes that have a nonzero degree. It follows that

$$|Q_n|\leq CV^4.$$

Moreover, $Q_n^*=\sum_{i,j,k,\ell(dist)}M_{ij}^*M_{jk}^*M_{k\ell}^*M_{\ell i}^*$, where $M_{ij}^*=\widetilde{\Omega}_{ij}+W_{ij}+\delta_{ij}$. Re-write $M_{ij}^*=A_{ij}-\eta_i^*\eta_j^*+\eta_i(\eta_j-\tilde{\eta}_j)+\eta_j(\eta_j-\tilde{\eta}_j)$. First, since $\eta_i^*\leq C\theta_i$ and $\eta_i\leq C\theta_i$ (see (81)), $|M_{ij}^*|\leq A_{ij}+C\theta_i\theta_j+C\theta_i|\eta_j-\tilde{\eta}_j|+C\theta_j|\eta_i-\tilde{\eta}_i|$. Second, note that $\tilde{\eta}_i$ equals to $v^{-1/2}$ times degree of node i, where $v\asymp\|\theta\|_1^2$ according to (80). It follows that $|\eta_i-\tilde{\eta}_i|\leq C(\theta_i+\|\theta\|_1^{-1}V)$. Therefore,

$$|M_{ij}^*|\leq A_{ij}+C\theta_i\theta_j+C\|\theta\|_1^{-1}V(\theta_i+\theta_j).$$

We plug it into the definition of Q_n^* and note that there are at most V pairs of (i,j) such that $A_{ij}\neq 0$. By elementary calculation,

$$|Q_n^*|\leq C(V^4+\|\theta\|_1^4).$$

Combining the above gives

$$(Q_n-Q_n^*)^2\leq 2Q_n^2+2(Q_n^*)^2\leq C(V^8+\|\theta\|_1^8). \tag{256}$$

Last, we show (253). By (256) and that $V^8\leq Cv^8+C|V-v|^8$, we have

$$\begin{aligned}\mathbb{E}[(Q_n-Q_n^*)^2\cdot I_{D^c}]&\leq C\mathbb{E}[|V-v|^8\cdot I_{D^c}]+C(v^8+\|\theta\|_1^8)\cdot\mathbb{P}(D^c)\\&\leq C\mathbb{E}[|V-v|^8\cdot I_{D^c}]+C\|\theta\|_1^{16}\cdot\mathbb{P}(D^c),\end{aligned} \tag{257}$$

where the second line is from $v\asymp\|\theta\|_1^2$. Note that $x_n\gg\sqrt{\log(\|\theta\|_1)}$. For n sufficiently large, $x_n^2\geq 17C_1^{-1}\log(\|\theta\|_1)$. Combining it with (255), we have

$$\|\theta\|_1^{16}\cdot\mathbb{P}(D^c)\leq\|\theta\|_1^{16}\cdot 2e^{-C_1x_n^2}\leq\|\theta\|_1^{16}\cdot 2e^{-17\|\theta\|_1}=o(1). \tag{258}$$

We then bound $\mathbb{E}[|V-v|^8\cdot I_{D^c}]$. Let $f(t)$ and $F(t)$ be the probability density and CDF of $|V-v|$, and write $\bar{F}(t)=1-F(t)$. Using integration by part, for any continuously differentiable function $g(t)$ and $x>0$, $\int_x^\infty g(t)f(t)dt=g(x)\bar{F}(x)+\int_x^\infty g'(t)\bar{F}(t)dt$. We apply the formula to $g(t)=t^8$ and $x=x_n\|\theta\|_1$. It yields

$$\begin{aligned}\mathbb{E}[|V-v|^8\cdot I_{D^c}]&=(x_n\|\theta\|_1)^8\cdot\mathbb{P}(D^c)+\int_{x_n\|\theta\|_1}^\infty 8t^7\cdot\mathbb{P}(|V-v|>t)dt\\&\equiv I+II.\end{aligned}$$

Consider I. By (258) and $x_n\ll\|\theta\|_1$,

$$I\ll\|\theta\|_1^{16}\cdot\mathbb{P}(D^c)=o(1).$$

Consider II. By (254), (258), and elementary probability,

$$\begin{aligned}II&\leq 8(\|\theta\|_1^2)^7\cdot\mathbb{P}\big(x_n\|\theta\|_1<|V-v|\leq\|\theta\|_1^2\big)+\int_{\|\theta\|_1^2}8t^7\cdot\mathbb{P}(|V-v|>t)dt\\&\leq C\|\theta\|_1^{14}\cdot\mathbb{P}(D^c)+\int_{\|\theta\|_1^2}8t^7\cdot 2e^{-C_2t}dt\\&=o(1),\end{aligned}$$

where in the last line we have used (258) and the fact that $\int_x^\infty t^7e^{-C_2t}dt\to 0$ as $x\to\infty$. Combining the bounds for I and II gives

$$\mathbb{E}[|V-v|^8\cdot I_{D^c}]=o(1). \tag{259}$$

Then, (253) follows by plugging (258)-(259) into (257).

TABLE G.4
The 34 types of the 175 post-expansion sums for $(\widetilde{Q}_n^* - Q_n^*)$.

Notation	#	$N_{\tilde{r}}$	$(N_\delta, N_{\widetilde{\Omega}}, N_W)$	Examples	N_W^*
R_1	4	1	(0, 0, 3)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} W_{jk} W_{k\ell} W_{\ell i}$	5
R_2	8	1	(0, 1, 2)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} W_{k\ell} W_{\ell i}$	4
R_3	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} W_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i}$	4
R_4	8	1	(0, 2, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i}$	3
R_5	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} W_{k\ell} \widetilde{\Omega}_{\ell i}$	3
R_6	4	1	(0, 3, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} \widetilde{\Omega}_{k\ell} \widetilde{\Omega}_{\ell i}$	2
R_7	8	1	(1, 0, 2)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} W_{k\ell} W_{\ell i}$	5
R_8	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} W_{jk} \delta_{k\ell} W_{\ell i}$	5
R_9	8	1	(1, 1, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i}$	4
R_{10}	8			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} W_{k\ell} \delta_{\ell i}$	4
R_{11}	8			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} W_{jk} \delta_{k\ell} \widetilde{\Omega}_{\ell i}$	4
R_{12}	8	1	(1, 2, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \widetilde{\Omega}_{k\ell} \widetilde{\Omega}_{\ell i}$	3
R_{13}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} \delta_{k\ell} \widetilde{\Omega}_{\ell i}$	3
R_{14}	8	1	(2, 0, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \delta_{k\ell} W_{\ell i}$	5
R_{15}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} W_{k\ell} \delta_{\ell i}$	5
R_{16}	8	1	(2, 1, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \delta_{k\ell} \widetilde{\Omega}_{\ell i}$	4
R_{17}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \widetilde{\Omega}_{k\ell} \delta_{\ell i}$	4
R_{18}	4	1	(3, 0, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \delta_{k\ell} \delta_{\ell i}$	5
R_{19}	4	2	(0, 0, 2)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} W_{k\ell} W_{\ell i}$	6
R_{20}	2			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} W_{jk} \tilde{r}_{k\ell} W_{\ell i}$	6
R_{21}	4	2	(0, 2, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \widetilde{\Omega}_{k\ell} \widetilde{\Omega}_{\ell i}$	4
R_{22}	2			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} \tilde{r}_{k\ell} \widetilde{\Omega}_{\ell i}$	4
R_{23}	4	2	(2, 0, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \delta_{k\ell} \delta_{\ell i}$	6
R_{24}	2			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \tilde{r}_{k\ell} \delta_{\ell i}$	6
R_{25}	8	2	(0, 1, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \widetilde{\Omega}_{k\ell} W_{\ell i}$	5
R_{26}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \widetilde{\Omega}_{jk} \tilde{r}_{k\ell} W_{\ell i}$	5
R_{27}	8	2	(1, 1, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \delta_{k\ell} \widetilde{\Omega}_{\ell i}$	5
R_{28}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \tilde{r}_{k\ell} \widetilde{\Omega}_{\ell i}$	5
R_{29}	8	2	(1, 0, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \delta_{k\ell} W_{\ell i}$	6
R_{30}	4			$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \delta_{jk} \tilde{r}_{k\ell} W_{\ell i}$	6
R_{31}	4	3	(0, 0, 1)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{k\ell} W_{\ell i}$	7
R_{32}	4	3	(0, 1, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{k\ell} \widetilde{\Omega}_{\ell i}$	6
R_{33}	4	3	(1, 0, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{k\ell} \delta_{\ell i}$	7
R_{34}	1	4	(0, 0, 0)	$\sum_{i,j,k,\ell(dist)} \tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{k\ell} \tilde{r}_{\ell i}$	8

G.4.10. *Proof of Lemma G.11.* There are 175 post-expansion sums in $(\widetilde{Q}_n^* - Q_n^*)$. They divide into 34 different types, denoted by R_1-R_{34} as shown in Table G.4. It suffices to prove that, for each $1 \le k \le 34$, under the null hypothesis,

(260) $$\big|\mathbb{E}[R_k]\big| = o(\|\theta\|^4), \qquad \mathrm{Var}(R_k) = o(\|\theta\|^8),$$

and under the alternative hypothesis,

(261) $$\big|\mathbb{E}[R_k]\big| = o(\alpha^4\|\theta\|^8), \qquad \mathrm{Var}(R_k) = O(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6).$$

We need some preparation. First, recall that $\tilde{r}_{ij} = -\frac{v}{V}(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)$. It follows that each post-expansion sum has the form

(262) $$\left(\frac{v}{V}\right)^{N_{\tilde{r}}} \sum_{i,j,k,\ell(dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i},$$

where a_{ij} takes values in $\{\widetilde{\Omega}_{ij}, W_{ij}, \delta_{ij}, -(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)\}$ and $b_{jk}, c_{k\ell}, d_{\ell i}$ are similar. The variable $\frac{v}{V}$ has a complicated correlation with each summand, so we want to get rid of it. Denote the variable in (262) by Y. Write $m = N_{\tilde{r}}$ and

$$Y = \Big(\frac{v}{V}\Big)^m X, \qquad \text{where} \quad X = \sum_{i,j,k,\ell(dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i}. \tag{263}$$

We compare the mean and variance of X and Y. By assumption, $\sqrt{\log(\|\theta\|_1)} \ll \|\theta\|_1/\|\theta\|^2$. Then, there exists a sequence x_n such that

$$\sqrt{\log(\|\theta\|_1)} \ll x_n \ll \|\theta\|_1/\|\theta\|^2, \qquad \text{as } n \to \infty.$$

We introduce an event

$$D = \big\{|V - v| \le \|\theta\|_1 x_n\big\}.$$

In Lemma G.10, we have proved $\mathbb{E}[(Q_n - Q_n^*)^2 \cdot I_{D^c}] = o(1)$. By similar proof, we can show: as long as $|Y - X|$ is bounded by a polynomial of V and $\|\theta\|_1$,

$$\mathbb{E}[(Y - X)^2 \cdot I_{D^c}] = o(1). \tag{264}$$

Additionally, on the event D, since $v \asymp \|\theta\|_1^2 \gg \|\theta\|_1 x_n$, we have $|V - v| = o(v)$. It follows that $\frac{|V-v|}{V} \lesssim \frac{|V-v|}{v} \le C\|\theta\|^{-1} x_n = o(1)$. For any fixed $m \ge 1$, $(1+x)^m \le 1 + Cx$ for x being close to 0. Hence, $|1 - \frac{v^m}{V^m}| \le C|1 - \frac{v}{V}| \le C\|\theta\|_1^{-1} x_n = o(\|\theta\|^{-2})$. It implies

$$|Y - X| = o(\|\theta\|^{-2}) \cdot |X|, \qquad \text{on the event } D. \tag{265}$$

By (264)-(265) and elementary probability,

$$\begin{aligned} |\mathbb{E}[Y - X]| &\le |\mathbb{E}[(Y - X) \cdot I_D]| + |\mathbb{E}[(Y - X) \cdot I_{D^c}]| \\ &\le o(\|\theta\|^{-2}) \cdot \mathbb{E}[|X| \cdot I_D] + \sqrt{\mathbb{E}[(Y - X)^2 \cdot I_{D^c}]} \\ &\le o(\|\theta\|^{-2})\sqrt{\mathbb{E}[X^2]} + o(1), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(Y) &\le 2\mathrm{Var}(X) + 2\mathrm{Var}(Y - X) \\ &\le 2\mathrm{Var}(X) + 2\mathbb{E}[(Y - X)^2] \\ &= 2\mathrm{Var}(X) + 2\mathbb{E}[(Y - X)^2 \cdot I_D] + 2\mathbb{E}[(Y - X)^2 \cdot I_{D^c}] \\ &\le 2\mathrm{Var}(X) + o(\|\theta\|^{-4}) \cdot \mathbb{E}[X^2] + o(1). \end{aligned}$$

Under the null hypothesis, suppose we can prove that

$$\mathbb{E}[X^2] = o(\|\theta\|^8). \tag{266}$$

Since $\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \mathrm{Var}(X)$, it implies $|\mathbb{E}[X]| = o(\|\theta\|^4)$ and $\mathrm{Var}(X) = o(\|\theta\|^8)$. Therefore,

$$\begin{aligned} &|\mathbb{E}[Y]| \le |\mathbb{E}[X]| + |\mathbb{E}[Y - X]| = o(\|\theta\|^4), \\ &\mathrm{Var}(Y) \le C\mathrm{Var}(X) + o(\|\theta\|^{-4}) \cdot \mathbb{E}[X^2] + o(1) = o(\|\theta\|^8). \end{aligned}$$

Under the alternative hypothesis, suppose we can prove that

$$|\mathbb{E}[X]| = O(\alpha^2\|\theta\|^6), \qquad \mathrm{Var}(X) = o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6). \tag{267}$$

Since $\mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \mathrm{Var}(X)$, we have $\mathbb{E}[X^2] = O(\alpha^4\|\theta\|^{12})$. Then,

$$|\mathbb{E}[Y]| \le O(\alpha^2\|\theta\|^6) + o(\|\theta\|^{-2}) \cdot O(\alpha^2\|\theta\|^6) = o(\alpha^4\|\theta\|^8),$$
$$\mathrm{Var}(Y) \le o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6) + o(\|\theta\|^{-4}) \cdot O(\alpha^4\|\theta\|^{12}) = o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6).$$

In conclusion, to prove that Y satisfies the requirement in (260)-(261), it is sufficient to prove that X satisfies (266)-(267). We remark that (267) puts a more stringent requirement on the mean of the variable, compared to (261).

From now on, in the analysis of each R_k of the form (262), we shall always neglect the factor $(\frac{v}{V})^{N_{\tilde{r}}}$, and show that, after this factor is removed, the random variable satisfies (266)-(267). This is equivalent to pretending

$$\tilde{r}_{ij} = -(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)$$

and proving each R_k satisfies (266)-(267). Unless mentioned, we stick to this mis-use of notation $\tilde{r}_{ij}$ in the proof below.

Second, we divide 34 terms into several groups using the *intrinsic order of* W defined below. Note that $\tilde{r}_{ij} = -(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)$, $\delta_{ij} = \eta_i(\eta_j - \tilde{\eta}_j) + \eta_j(\eta_i - \tilde{\eta}_i)$, and $\tilde{\eta}_i - \eta_i = \frac{1}{\sqrt{v}}\sum_{s\neq i} W_{is}$. We thus have

$$\tilde{r}_{ij} = -\frac{1}{v}\Big(\sum_{s\neq i} W_{is}\Big)\Big(\sum_{t\neq j} W_{jt}\Big), \qquad \delta_{ij} = -\frac{1}{\sqrt{v}}\eta_i\Big(\sum_{t\neq j} W_{jt}\Big) - \frac{1}{\sqrt{v}}\eta_j\Big(\sum_{s\neq i} W_{is}\Big).$$

Each $\tilde{r}_{ij}$ is a weighted sum of terms like $W_{is}W_{jt}$, and each δ_{ij} is a weighted sum of terms like W_{jt}. Intuitively, we view $\tilde{r}$-term as an "order-2 W-term" and view δ-term as "order-1 W-term." It motivates the definition of *intrinsic order of* W as

$$N_W^* = N_W + N_\delta + 2N_{\tilde{r}}. \tag{268}$$

We group 34 terms by the value of N_W^*; see the last column of Table G.4.

G.4.10.1. Analysis of post-expansion sums with $N_W^* \le 4$. There are 14 such terms, including R_2-R_6, R_9-R_{13}, R_{16}-R_{17}, and R_{21}-R_{22}. They all equal to zero under the null hypothesis, so it is sufficient to show that they satisfy (267) under the alternative hypothesis. We prove by comparing each R_k to some previously analyzed terms. Take R_9 for example. Plugging in the definition of $\tilde{r}_{ij}$ and δ_{ij} gives

$$\begin{aligned} R_9 &= \sum_{i,j,k,\ell (dist)} [(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)][(\tilde{\eta}_j - \eta_j)\eta_k + \eta_j(\tilde{\eta}_k - \eta_k)]\widetilde{\Omega}_{k\ell}W_{\ell i} \\ &= R_{9a} + R_{9b}, \end{aligned}$$

where

$$R_{9a} = \sum_{i,j,k,\ell (dist)} \eta_k \widetilde{\Omega}_{k\ell} \cdot [(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)^2 W_{\ell i}],$$
$$R_{9b} = \sum_{i,j,k,\ell (dist)} \eta_j \widetilde{\Omega}_{k\ell} \cdot [(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)(\tilde{\eta}_k - \eta_k) W_{\ell i}]. \tag{269}$$

At the same time, we recall that T_1 in Lemmas G.8-G.9 is defined as

$$T_1 = \sum_{i,j,k,\ell (dist)} \delta_{ij}\delta_{jk}\delta_{k\ell}W_{\ell i} = \sum_{i,j,k,\ell (dist)} \delta_{\ell j}\delta_{jk}\delta_{ki}W_{i\ell}.$$

In the proof of the above two lemmas, we express T_1 as the weighted sum of T_{1a}-T_{1d}; see (130). Note that T_{1a} and T_{1d} in (130) can be re-written as

$$\begin{aligned} T_{1d} &= \sum_{i,j,k,\ell(dist)} [\eta_\ell(\tilde{\eta}_j - \eta_j)][(\tilde{\eta}_j - \eta_j)\eta_k][\eta_k(\tilde{\eta}_i - \eta_i)]W_{i\ell} \\ &= \sum_{i,j,k,\ell(dist)} \eta_k^2\eta_\ell \cdot [(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)^2 W_{\ell i}], \\ T_{1a} &= \sum_{i,j,k,\ell(dist)} [\eta_\ell(\tilde{\eta}_j - \eta_j)][\eta_j(\tilde{\eta}_k - \eta_k)][\eta_k(\tilde{\eta}_i - \eta_i)]W_{i\ell} \\ &= \sum_{i,j,k,\ell(dist)} \eta_j\eta_k\eta_\ell \cdot [(\tilde{\eta}_i - \eta_i)(\tilde{\eta}_j - \eta_j)(\tilde{\eta}_k - \eta_k)W_{i\ell}]. \end{aligned} \tag{270}$$

Compare (269) and (270). It is seen that R_{9a} and T_{1d} have the same structure, where the non-stochastic coefficients in the summand satisfy $|\eta_k\widetilde{\Omega}_{k\ell}| \leq C\alpha\theta_k^2\theta_\ell$ and $|\eta_k^2\eta_\ell| \leq C\theta_k^2\theta_\ell$, respectively. This means we can bound $|\mathbb{E}(R_{9a})|$ and $\mathrm{Var}(R_{9a})$ in the same way as we bound $|\mathbb{E}[T_{1d}]|$ and $\mathrm{Var}(T_{1d})$, and the bounds have an extra factor of α and α^2, respectively. In detail, in the proof of Lemmas G.8-G.9, we have shown

$$|\mathbb{E}[T_{1d}]| \leq C\|\theta\|^4, \qquad \mathrm{Var}(T_{1d}) \leq \frac{C\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1}.$$

It follows immediately that

$$|\mathbb{E}[R_{9a}]| \leq C\alpha\|\theta\|^4 = o(\alpha^2\|\theta\|^6), \qquad \mathrm{Var}(T_{1d}) \leq \frac{C\alpha^2\|\theta\|^6\|\theta\|_3^3}{\|\theta\|_1} = o(\|\theta\|^8).$$

Similarly, since we have proved

$$|\mathbb{E}[T_{1a}]| \leq \frac{C\|\theta\|^6}{\|\theta\|_1^2}, \qquad \mathrm{Var}(T_{1a}) \leq \frac{C\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2},$$

it follows immediately that

$$|\mathbb{E}[R_{9b}]| \leq \frac{C\alpha\|\theta\|^6}{\|\theta\|_1^2} = o(\alpha^2\|\theta\|^6), \qquad \mathrm{Var}(R_{9b}) \leq \frac{C\alpha^2\|\theta\|^4\|\theta\|_3^6}{\|\theta\|_1^2} = o(\|\theta\|^8).$$

This proves (267) for $X = R_{9a}$.

We use the same strategy to bound all other terms with $N_W^* \leq 4$. The details are in Table G.5. In each row of the table, the left column displays a targeting variable X, and the right column displays a previously analyzed variable, which we call X^*, that has a similar structure as X. It is not hard to see that we can obtain upper bounds for $|\mathbb{E}[X]|$ and $\mathrm{Var}(X)$ from multiplying the upper bounds of $|\mathbb{E}[X^*]|$ and $\mathrm{Var}(X^*)$ by α^m and α^{2m}, respectively, where m is a nonnegative integer (e.g., $m = 1$ in the analysis of R_9). Using our previous results, each X^* in the right column satisfies

$$|\mathbb{E}[X^*]| = O(\alpha^2\|\theta\|^6), \qquad \mathrm{Var}(X^*) = o(\|\theta\|^8 + \alpha^6\|\theta\|^8\|\theta\|_3^6).$$

So, each X in the left column satisfies (267).

G.4.10.2. Analysis of post-expansion sums with $N_W^* = 5$. There are 10 such terms, including R_1, R_7-R_8, R_{14}-R_{15}, R_{18}, and R_{25}-R_{28}. Using the the notation

$$G_i = \tilde{\eta}_i - \eta_i,$$

TABLE G.5
The 14 types of post-expansion sums with $N_W^* \le 4$. The right column displays the post-expansion sums defined before which have similar forms as the post-expansion sums in the left column. Definitions of the terms in the right column can be found in (94), (100), (106), (116), (122), (130), (131), and (132). For some terms in the right column, we permute (i,j,k,ℓ) in the original definition for ease of comparison with the left column. (In all expressions, the subscript "$i,j,k,\ell(dist)$" is omitted.)

	Expression		Expression
R_2	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}W_{k\ell}W_{\ell i}$	Z_{1b}	$\sum(\tilde\eta_i-\eta_i)\eta_j(\tilde\eta_j-\eta_j)\eta_k W_{k\ell}W_{\ell i}$
R_3	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)W_{jk}\widetilde\Omega_{k\ell}W_{\ell i}$	Z_{2a}	$\sum\eta_\ell(\tilde\eta_j-\eta_j)W_{jk}\eta_k(\tilde\eta_i-\eta_i)W_{i\ell}$
R_4	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}\widetilde\Omega_{k\ell}W_{\ell i}$	Z_{3d}	$\sum(\tilde\eta_i-\eta_i)\eta_j(\tilde\eta_j-\eta_j)\eta_k\widetilde\Omega_{k\ell}W_{\ell i}$
R_5	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}W_{k\ell}\widetilde\Omega_{\ell i}$	Z_{4b}	$\sum\widetilde\Omega_{ij}(\tilde\eta_j-\eta_j)\eta_k W_{k\ell}\eta_\ell(\tilde\eta_i-\eta_i)$
R_6	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}$	Z_{5a}	$\sum\eta_i(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}\widetilde\Omega_{k\ell}\eta_\ell(\tilde\eta_i-\eta_i)$
R_9	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2\eta_k\widetilde\Omega_{k\ell}W_{\ell i}$	T_{1d}	$\sum\eta_\ell(\tilde\eta_j-\eta_j)^2\eta_k^2(\tilde\eta_i-\eta_i)W_{i\ell}$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\widetilde\Omega_{k\ell}W_{\ell i}$	T_{1a}	$\sum\eta_\ell(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\eta_k(\tilde\eta_i-\eta_i)W_{i\ell}$
R_{10}	$\sum(\tilde\eta_i-\eta_i)^2(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}W_{k\ell}\eta_\ell$	T_{1c}	$\sum(\tilde\eta_j-\eta_j)\eta_k W_{k\ell}\eta_\ell(\tilde\eta_i-\eta_i)^2\eta_j$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}W_{k\ell}(\tilde\eta_\ell-\eta_\ell)\eta_i$	T_{1a}	$\sum(\tilde\eta_j-\eta_j)\eta_k W_{k\ell}(\tilde\eta_\ell-\eta_\ell)\eta_i(\tilde\eta_i-\eta_i)\eta_j$
R_{11}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)W_{jk}\eta_k(\tilde\eta_\ell-\eta_\ell)\widetilde\Omega_{\ell i}$	T_{1a}	$\sum(\tilde\eta_i-\eta_i)\eta_k W_{kj}(\tilde\eta_j-\eta_j)\eta_\ell(\tilde\eta_\ell-\eta_\ell)\eta_i$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)W_{jk}(\tilde\eta_k-\eta_k)\eta_\ell\widetilde\Omega_{\ell i}$	T_{1b}	$\sum\eta_i(\tilde\eta_k-\eta_k)W_{kj}(\tilde\eta_j-\eta_j)\eta_\ell^2(\tilde\eta_i-\eta_i)$
R_{12}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2\eta_k\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}$	T_{2c}	$\sum\eta_i(\tilde\eta_j-\eta_j)^2\eta_k\widetilde\Omega_{k\ell}\eta_\ell(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}$	T_{2a}	$\sum\eta_i(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\widetilde\Omega_{k\ell}\eta_\ell(\tilde\eta_i-\eta_i)$
R_{13}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}(\tilde\eta_k-\eta_k)\eta_\ell\widetilde\Omega_{\ell i}$	T_{2b}	$\sum\eta_i(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}(\tilde\eta_k-\eta_k)\eta_\ell^2(\tilde\eta_i-\eta_i)$
R_{16}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2\eta_k(\tilde\eta_k-\eta_k)\eta_\ell\widetilde\Omega_{\ell i}$	F_b	$\sum\eta_i(\tilde\eta_j-\eta_j)^2\eta_k(\tilde\eta_k-\eta_k)\eta_\ell^2(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2\eta_k^2(\tilde\eta_\ell-\eta_\ell)\widetilde\Omega_{\ell i}$	F_b	$\sum\eta_i(\tilde\eta_j-\eta_j)^2\eta_k^2(\tilde\eta_\ell-\eta_\ell)\eta_\ell(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)^2\eta_\ell\widetilde\Omega_{\ell i}$	F_b	$\sum\eta_i(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)^2\eta_\ell^2(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\eta_k(\tilde\eta_\ell-\eta_\ell)\widetilde\Omega_{\ell i}$	F_a	$\sum\eta_i(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\eta_k(\tilde\eta_\ell-\eta_\ell)\eta_\ell(\tilde\eta_i-\eta_i)$
R_{17}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\widetilde\Omega_{k\ell}(\tilde\eta_\ell-\eta_\ell)\eta_i$	F_a	$\sum\eta_i(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\eta_k(\tilde\eta_\ell-\eta_\ell)\eta_\ell(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2\eta_k\widetilde\Omega_{k\ell}(\tilde\eta_\ell-\eta_\ell)\eta_i$	F_b	$\sum\eta_i(\tilde\eta_j-\eta_j)^2\eta_k^2(\tilde\eta_\ell-\eta_\ell)\eta_\ell(\tilde\eta_i-\eta_i)$
	$\sum(\tilde\eta_i-\eta_i)^2(\tilde\eta_j-\eta_j)^2\eta_k\widetilde\Omega_{k\ell}\eta_\ell$	F_c	$\sum\eta_\ell(\tilde\eta_i-\eta_i)^2\eta_k^2(\tilde\eta_j-\eta_j)^2\eta_\ell$
R_{21}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)^2(\tilde\eta_k-\eta_k)\widetilde\Omega_{k\ell}\widetilde\Omega_{\ell i}$	F_b	$\sum\eta_i(\tilde\eta_j-\eta_j)^2\eta_k(\tilde\eta_k-\eta_k)\eta_\ell^2(\tilde\eta_i-\eta_i)$
R_{22}	$\sum(\tilde\eta_i-\eta_i)(\tilde\eta_j-\eta_j)\widetilde\Omega_{jk}(\tilde\eta_k-\eta_k)(\tilde\eta_\ell-\eta_\ell)\widetilde\Omega_{\ell i}$	F_a	$\sum\eta_i(\tilde\eta_j-\eta_j)\eta_j(\tilde\eta_k-\eta_k)\eta_k(\tilde\eta_\ell-\eta_\ell)\eta_\ell(\tilde\eta_i-\eta_i)$

we get the following expressions (note: factors of $(\frac{v}{V})^m$ have been removed; see explanations in (266)-(267)):

$$R_1 = \sum_{i,j,k,\ell(dist)} G_iG_jW_{jk}W_{k\ell}W_{\ell i},$$

$$\begin{aligned} R_7 &= \sum_{i,j,k,\ell(dist)} G_iG_j\eta_jG_kW_{k\ell}W_{\ell i} + \sum_{i,j,k,\ell(dist)} G_iG_j^2\eta_kW_{k\ell}W_{\ell i} \\ &= \sum_{i,j,k,\ell(dist)} \eta_j(G_iG_jG_kW_{k\ell}W_{\ell i}) + \sum_{i,j,k,\ell(dist)} \eta_k(G_iG_j^2W_{k\ell}W_{\ell i}), \end{aligned}$$

$$R_8 = 2\sum_{i,j,k,\ell(dist)} G_iG_jW_{jk}\eta_kG_\ell W_{\ell i} = 2\sum_{i,j,k,\ell(dist)} \eta_k(G_iG_jG_\ell W_{jk}W_{\ell i}),$$

$$\begin{aligned} R_{14} &= \sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j^2\eta_k^2G_\ell W_{\ell i} + 2\sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j^2\eta_kG_k\eta_\ell W_{\ell i} + \sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j\eta_jG_k\eta_kG_\ell W_{\ell i} \\ &= \sum_{\substack{i,j,k,\ell\\(dist)}} \eta_k^2(G_iG_j^2G_\ell W_{\ell i}) + 2\sum_{\substack{i,j,k,\ell\\(dist)}} \eta_k\eta_\ell(G_iG_j^2G_kW_{\ell i}) + \sum_{\substack{i,j,k,\ell\\(dist)}} \eta_j\eta_k(G_iG_jG_kG_\ell W_{\ell i}), \end{aligned}$$

$$R_{15}=\sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j\eta_jG_kW_{k\ell}G_\ell\eta_i+2\sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j^2\eta_kW_{k\ell}G_\ell\eta_i+\sum_{\substack{i,j,k,\ell\\(dist)}} G_iG_j^2\eta_kW_{k\ell}\eta_\ell G_i$$

$$=\sum_{\substack{i,j,k,\ell\\(dist)}} \eta_i\eta_j(G_iG_jG_kG_\ell W_{k\ell})+2\sum_{\substack{i,j,k,\ell\\(dist)}} \eta_i\eta_k(G_iG_j^2G_\ell W_{k\ell})+\sum_{\substack{i,j,k,\ell\\(dist)}} \eta_k\eta_\ell(G_i^2G_j^2W_{k\ell}),$$

$$R_{18}=4\sum_{i,j,k,\ell(dist)} \eta_j\eta_k\eta_\ell(G_i^2G_jG_kG_\ell)+4\sum_{i,j,k,\ell(dist)} \eta_k\eta_\ell^2(G_i^2G_j^2G_k),$$

$$R_{25}=\sum_{i,j,k,\ell(dist)} G_iG_j^2G_k\widetilde{\Omega}_{k\ell}W_{\ell i}=\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{k\ell}(G_iG_j^2G_kW_{\ell i}),$$

$$R_{26}=\sum_{i,j,k,\ell(dist)} G_iG_j\widetilde{\Omega}_{jk}G_kG_\ell W_{\ell i}=\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{jk}(G_iG_jG_kG_\ell W_{\ell i}),$$

$$R_{27}=\sum_{i,j,k,\ell(dist)} G_iG_j^2G_k\eta_kG_\ell\widetilde{\Omega}_{\ell i}+\sum_{i,j,k,\ell(dist)} G_iG_j^2G_k^2\eta_\ell\widetilde{\Omega}_{\ell i}$$

$$=\sum_{i,j,k,\ell(dist)} \eta_k\widetilde{\Omega}_{\ell i}(G_iG_j^2G_kG_\ell)+\sum_{i,j,k,\ell(dist)} \eta_\ell\widetilde{\Omega}_{\ell i}(G_iG_j^2G_k^2),$$

$$R_{28}=2\sum_{i,j,k,\ell(dist)} G_iG_j\eta_jG_k^2G_\ell\widetilde{\Omega}_{\ell i}=2\sum_{i,j,k,\ell(dist)} \eta_j\widetilde{\Omega}_{\ell i}(G_iG_jG_k^2G_\ell).$$

Each expression above belongs to one of the following types:

$$J_1=\sum_{i,j,k,\ell(dist)} G_iG_jW_{jk}W_{k\ell}W_{\ell i},\qquad J_2=\sum_{i,j,k,\ell(dist)} \eta_j(G_iG_jG_kW_{k\ell}W_{\ell i}),$$

$$J_3=\sum_{i,j,k,\ell(dist)} \eta_k(G_iG_jG_\ell W_{jk}W_{\ell i}),\qquad J_4=\sum_{i,j,k,\ell(dist)} \eta_k(G_iG_j^2W_{k\ell}W_{\ell i}),$$

$$J_5=\sum_{i,j,k,\ell(dist)} \eta_j\eta_k(G_iG_jG_kG_\ell W_{\ell i}),\qquad J_5'=\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{jk}(G_iG_jG_kG_\ell W_{\ell i}),$$

$$J_6=\sum_{i,j,k,\ell(dist)} \eta_k\eta_\ell(G_iG_j^2G_kW_{\ell i}),\qquad J_6'=\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{k\ell}(G_iG_j^2G_kW_{\ell i}),$$

$$J_7=\sum_{i,j,k,\ell(dist)} \eta_k^2(G_iG_j^2G_\ell W_{\ell i}),\qquad J_8=\sum_{i,j,k,\ell(dist)} \eta_k\eta_\ell(G_i^2G_j^2W_{k\ell}),$$

$$J_9=\sum_{i,j,k,\ell(dist)} \eta_k\widetilde{\Omega}_{\ell i}(G_iG_j^2G_kG_\ell),\qquad J_{10}=\sum_{i,j,k,\ell(dist)} \eta_\ell\widetilde{\Omega}_{\ell i}(G_iG_j^2G_k^2).$$

Since $|\eta_j\eta_k|\leq C\theta_j\theta_k$ and $|\widetilde{\Omega}_{jk}|\leq C\alpha\theta_j\theta_k$, the study of J_5 and J_5' are similar. Also, the study of J_6 and J_6' are similar. We now study J_1-J_{10}. Consider J_1. It is seen that

$$J_1=\frac{1}{v}\sum_{i,j,k,\ell(dist)}\Big(\sum_{s\neq i}W_{is}\Big)\Big(\sum_{t\neq j}W_{jt}\Big)W_{jk}W_{k\ell}W_{\ell i}=\frac{1}{v}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j}} W_{is}W_{i\ell}W_{jt}W_{jk}W_{k\ell}.$$

Since s can be equal to ℓ and t can be equal to k, there are three different types:

$$J_{1a} = \frac{1}{v} \sum_{i,j,k,\ell(dist)} W_{i\ell}^2 W_{jk}^2 W_{k\ell}, \qquad J_{1b} = \frac{1}{v} \sum_{\substack{i,j,k,\ell(dist)\\ t\notin\{j,k\}}} W_{i\ell}^2 W_{jt} W_{jk} W_{k\ell},$$

$$J_{1c} = \frac{1}{v} \sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{i,\ell\},t\notin\{j,k\}}} W_{is} W_{i\ell} W_{jt} W_{jk} W_{k\ell}.$$

We now calculate $\mathbb{E}[J_{1a}^2]$-$\mathbb{E}[J_{1c}^2]$. Take J_{1a} for example. In order to get nonzero $\mathbb{E}[W_{i\ell}^2 W_{jk}^2 W_{k\ell} W_{i'\ell'}^2 W_{j'k'}^2 W_{k'\ell'}]$, we need either $W_{k\ell} = W_{k'\ell'}$ or each of the two variables $(W_{k\ell}, W_{k',\ell'})$ equals to another squared-W term. The leading term of $\mathbb{E}[J_{1a}^2]$ comes from the first case. In this case, we have $W_{k\ell} = W_{k'\ell'}$ but allow for $W_{i\ell} \neq W_{i'\ell'}$ and $W_{jk} \neq W_{j'k'}$. It has to be the case of either $(k',\ell') = (k,\ell)$ or $(k',\ell') = (\ell,k)$. Therefore, we have $\mathbb{E}[W_{i\ell}^2 W_{jk}^2 W_{k\ell} W_{i'\ell'}^2 W_{j'k'}^2 W_{k'\ell'}] = \mathbb{E}[W_{i\ell}^2 W_{jk}^2 W_{i'\ell'}^2 W_{j'k'}^2 W_{k\ell}^2]$. Using similar arguments, we have the following results, where details are omitted, as they are similar to the calculations in the proof of Lemmas G.4-G.9.

$$\mathbb{E}[J_{1a}^2] \leq \frac{C}{v^2} \sum_{\substack{i,j,k,\ell\\ i',j'}} \mathbb{E}[W_{i\ell}^2 W_{jk}^2 W_{i'\ell}^2 W_{j'k}^2 W_{k\ell}^2] \leq \frac{C}{\|\theta\|_1^4} \sum_{\substack{i,j,k,\ell\\ i',j'}} \theta_i\theta_j\theta_k^3\theta_\ell^3\theta_{i'}\theta_{j'} \leq C\|\theta\|_3^6,$$

$$\mathbb{E}[J_{1b}^2] \leq \frac{C}{v^2} \sum_{\substack{i,j,k,\ell,t\\ i'}} \mathbb{E}[W_{i\ell}^2 W_{i'\ell}^2 W_{jt}^2 W_{jk}^2 W_{k\ell}^2] \leq \frac{C}{\|\theta\|_1^4} \sum_{\substack{i,j,k,\ell,t\\ i'}} \theta_i\theta_j^2\theta_k^2\theta_\ell^3\theta_t\theta_{i'} \leq \frac{C\|\theta\|^4\|\theta\|_3^3}{\|\theta\|_1},$$

$$\mathbb{E}[J_{1c}^2] \leq \frac{C}{v^2} \sum_{i,j,k,\ell,s,t} \mathbb{E}[W_{is}^2 W_{i\ell}^2 W_{jt}^2 W_{jk}^2 W_{k\ell}^2] \leq \frac{C}{\|\theta\|_1^4} \sum_{i,j,k,\ell,s,t} \theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s\theta_t \leq \frac{C\|\theta\|^8}{\|\theta\|_1^2}.$$

The right hand sides are all $o(\|\theta\|^8)$. It follows that

$$\mathbb{E}[J_1^2] = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider J_2-J_4. By definition,

$$J_2 = \frac{1}{v\sqrt{v}} \sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j,q\neq k}} \eta_j W_{is} W_{jt} W_{kq} W_{k\ell} W_{\ell i}, \qquad J_3 = \frac{1}{v\sqrt{v}} \sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j,q\neq \ell}} \eta_k W_{is} W_{jt} W_{\ell q} W_{jk} W_{\ell i},$$

$$J_4 = \frac{1}{v\sqrt{v}} \sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j,q\neq j}} \eta_k W_{is} W_{jt} W_{jq} W_{k\ell} W_{\ell i}.$$

The analysis is summarized in Table G.6. In the first column of this table, we study different types of summands. For example, in the expression of J_2, $W_{is}W_{kq}W_{k\ell}W_{\ell i}$ have four different cases: (a) $W_{k\ell}^2 W_{\ell i}^2$, (b) $W_{k\ell}^2 W_{\ell i} W_{is}$ or $W_{k\ell} W_{\ell i}^2 W_{kq}$, (c) $W_{k\ell} W_{\ell i} W_{ik}^2$, and (d) $W_{k\ell} W_{\ell i} W_{is} W_{kq}$. In cases (b) and (d), W_{is} or W_{kq} may further equal to W_{jt}. Having explored all variants and considered index symmetry, we end up with 6 different cases, as listed in the first column of Table G.6. In the second column, we study the mean of the squares of the sum of each type of summands. Take the first row for example. We aim to study

$$\mathrm{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ t\neq j}} \eta_j (W_{k\ell}^2 W_{\ell i}^2) W_{jt}\Big)\Big].$$

The naive expansion gives the sum of $\eta_j\eta_{j'}\,\mathbb{E}[W_{k\ell}^2 W_{\ell i}^2 W_{jt} W_{k'\ell'}^2 W_{\ell' i'}^2 W_{j't'}]$ over $(i,j,k,\ell,t,i',j',k',\ell',t')$. However, for this term to be nonzero, all single-W terms have to be paired (either with another single-W term or with a squared-W term). The main contribution is from the case of $W_{jt} = W_{j't'}$. This is satisfied only when $(j',s') = (j,s)$ or $(j',s') = (s,j)$. By calculations which are omitted here, we can show that $(j',s') = (j,s)$ yields a larger bound. Therefore, it reduces to the sum of $\eta_j^2\,\mathbb{E}[(W_{jt}^2) W_{k\ell}^2 W_{\ell i}^2 W_{k'\ell'}^2 W_{\ell' i'}^2]$ over $(i,j,k,\ell,t,i',k',\ell')$, which is displayed in the second column of the table. In the last column, we sum the quantity in the second column over indices; it gives rise to a bound for the mean of the square of sum. See the table for details. Recall that the definition of J_2-J_4 contains a factor of $\frac{1}{v\sqrt{v}}$ in front of the sum, where $v \asymp \|\theta\|_1^2$ by (80). Hence, to get a desired bound, we only need that each row in the third column of Table G.6 is

$$o(\|\theta\|^8\|\theta\|_1^6).$$

This is true. We thus conclude that

$$\max\left\{\mathbb{E}[J_2^2],\,\mathbb{E}[J_3^2],\,\mathbb{E}[J_4^2]\right\} = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

TABLE G.6

Analysis of J_2-J_4. In the second column, the variables in brackets are paired W terms.

	Types of summand	Terms in mean-squared	Bound
J_2	$\eta_j(W_{k\ell}^2 W_{\ell i}^2)W_{jt}$	$\eta_j^2\,\mathbb{E}[(W_{jt}^2)W_{k\ell}^2 W_{\ell i}^2 W_{k'\ell'}^2 W_{\ell' i'}^2] \le \theta_i\theta_j^3\theta_k\theta_\ell^2\theta_t\theta_{i'}\theta_{k'}\theta_{\ell'}^2$	$\|\theta\|^4\|\theta\|_3^3\|\theta\|_1^5$
	$\eta_j(W_{k\ell}W_{\ell i}W_{ik}^2)W_{jt}$	$\eta_j^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{jt}^2)W_{ik}^4] \le C\theta_i^2\theta_j^3\theta_k^2\theta_\ell^2\theta_t$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1$
	$\eta_j(W_{k\ell}^2 W_{\ell i}W_{is})W_{jt}$	$\eta_j^2\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{jt}^2)W_{k\ell}^2 W_{k'\ell}^2] \le C\theta_i^2\theta_j^3\theta_k\theta_\ell^3\theta_s\theta_t\theta_{k'}$	$\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^4$
	$\eta_j(W_{k\ell}^2 W_{\ell i})W_{ij}^2$	$\eta_j\eta_{j'}\,\mathbb{E}[(W_{\ell i}^2)W_{k\ell}^2 W_{ij}^2 W_{k'\ell}^2 W_{ij'}^2] \le C\theta_i^3\theta_j^2\theta_k\theta_\ell^3\theta_{j'}^2\theta_{k'}$	$\|\theta\|^4\|\theta\|_3^6\|\theta\|_1^2$
	$\eta_j(W_{k\ell}W_{\ell i}W_{kq}W_{is})W_{jt}$	$\eta_j^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{kq}^2 W_{is}^2 W_{jt}^2)] \le C\theta_i^2\theta_j^3\theta_k^2\theta_\ell^2\theta_s\theta_t\theta_q$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1^3$
	$\eta_j(W_{k\ell}W_{\ell i})W_{kq}W_{ij}^2$	$\eta_j\eta_{j'}\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{kq}^2)W_{ij}^2 W_{ij'}^2] \le C\theta_i^3\theta_j^2\theta_k^2\theta_\ell^2\theta_q\theta_{j'}^2$	$\|\theta\|^8\|\theta\|_3^3\|\theta\|_1$
J_3	$\eta_k W_{\ell i}^3 W_{jk}^2$	$\eta_k\eta_{k'}\,\mathbb{E}[W_{\ell i}^3 W_{jk}^2 W_{\ell' i'}^3 W_{j'k'}^2] \le C\theta_i\theta_j\theta_k^2\theta_\ell\theta_{i'}\theta_{j'}\theta_{k'}^2\theta_{\ell'}$	$\|\theta\|^4\|\theta\|_1^6$
	$\eta_k W_{\ell i}^3(W_{jk}W_{jt})$	$\eta_k^2\,\mathbb{E}[(W_{jk}^2 W_{jt}^2)W_{\ell i}^3 W_{\ell' i'}^3] \le C\theta_i\theta_j^2\theta_k^3\theta_\ell\theta_t\theta_{i'}\theta_{\ell'}$	$\|\theta\|^2\|\theta\|_3^3\|\theta\|_1^5$
	$\eta_k(W_{\ell i}^2 W_{is})W_{jk}^2$	$\eta_k\eta_{k'}\,\mathbb{E}[(W_{is}^2)W_{\ell i}^2 W_{jk}^2 W_{\ell' i}^2 W_{j'k'}^2] \le C\theta_i^3\theta_j\theta_k^2\theta_\ell\theta_s\theta_{j'}\theta_{k'}^2\theta_{\ell'}$	$\|\theta\|^4\|\theta\|_3^3\|\theta\|_1^5$
	$\eta_k(W_{\ell i}^2 W_{is})W_{jk}W_{jt}$	$\eta_k^2\,\mathbb{E}[(W_{is}^2 W_{jk}^2 W_{jt}^2)W_{\ell i}^2 W_{\ell' i}^2] \le C\theta_i^3\theta_j^2\theta_k^3\theta_\ell\theta_s\theta_t\theta_{\ell'}$	$\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^4$
	$\eta_k W_{\ell i}^2 W_{ij}^2 W_{jk}$	$\eta_k^2\,\mathbb{E}[(W_{jk}^2)W_{\ell i}^2 W_{ij}^2 W_{\ell' i'}^2 W_{i'j}^2] \le C\theta_i^2\theta_j^3\theta_k^3\theta_\ell\theta_{i'}^2\theta_{\ell'}$	$\|\theta\|^4\|\theta\|_3^6\|\theta\|_1^2$
	$\eta_k(W_{\ell i}W_{is}W_{\ell q})W_{jk}^2$	$\eta_k\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell q}^2)W_{jk}^2 W_{j'k'}^2] \le C\theta_i^2\theta_j\theta_k^2\theta_\ell^2\theta_s\theta_q\theta_{j'}\theta_{k'}^2$	$\|\theta\|^8\|\theta\|_1^4$
	$\eta_k(W_{\ell i}W_{is}W_{\ell q})W_{jk}W_{jt}$	$\eta_k^2\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell q}^2 W_{jk}^2 W_{jt}^2)] \le C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^2\theta_s\theta_t\theta_q$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1^3$
	$\eta_k W_{\ell i}W_{ij}^2 W_{\ell q}W_{jk}$	$\eta_k^2\,\mathbb{E}[(W_{\ell i}^2 W_{\ell q}^2 W_{jk}^2)W_{ij}^4] \le C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^2\theta_q$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1$
J_4	$\eta_k(W_{k\ell}W_{\ell i}^2)W_{jt}^2$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2)W_{\ell i}^2 W_{jt}^2 W_{\ell i'}^2 W_{j't'}^2] \le C\theta_i\theta_j\theta_k^3\theta_\ell^3\theta_t\theta_{i'}\theta_{j'}\theta_{t'}$	$\|\theta\|_3^6\|\theta\|_1^6$
	$\eta_k(W_{k\ell}W_{\ell i}^2)W_{jt}W_{jq}$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2 W_{jt}^2 W_{jq}^2)W_{\ell i}^2 W_{\ell i'}^2] \le C\theta_i\theta_j^2\theta_k^3\theta_\ell^3\theta_t\theta_q\theta_{i'}$	$\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^4$
	$\eta_k(W_{k\ell}W_{\ell i}W_{is})W_{jt}^2$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{is}^2)W_{jt}^2 W_{j't'}^2] \le C\theta_i^2\theta_j\theta_k^3\theta_\ell^2\theta_s\theta_t\theta_{j'}\theta_{t'}$	$\|\theta\|^4\|\theta\|_3^3\|\theta\|_1^5$
	$\eta_k W_{k\ell}W_{\ell i}W_{ij}^3$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2)W_{ij}^3 W_{ij'}^3] \le C\theta_i^3\theta_j\theta_k^3\theta_\ell^2\theta_{j'}$	$\|\theta\|^2\|\theta\|_3^6\|\theta\|_1^2$
	$\eta_k(W_{k\ell}W_{\ell i}W_{is})W_{jt}W_{jq}$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{is}^2 W_{jt}^2 W_{jq}^2] \le C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^2\theta_s\theta_t\theta_q$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1^3$
	$\eta_k W_{k\ell}W_{\ell i}W_{ij}^2 W_{jq}$	$\eta_k^2\,\mathbb{E}[(W_{k\ell}^2 W_{\ell i}^2 W_{jq}^2)W_{ij}^4] \le C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^2\theta_q$	$\|\theta\|^6\|\theta\|_3^3\|\theta\|_1$

Consider J_5-J_8. It is seen that

$$J_5 = \frac{1}{v^2}\sum_{i,j,k,\ell(dist)} \eta_j\eta_k W_{is}W_{jt}W_{kq}W_{\ell m}W_{\ell i}, \quad J_6 = \frac{1}{v^2}\sum_{i,j,k,\ell(dist)} \eta_k\eta_\ell W_{is}W_{jt}W_{jq}W_{km}W_{\ell i},$$

$$J_7 = \frac{1}{v^2}\sum_{i,j,k,\ell(dist)} \eta_k^2 W_{is}W_{jt}W_{jq}W_{\ell m}W_{\ell i}, \qquad J_8 = \frac{1}{v^2}\sum_{i,j,k,\ell(dist)} \eta_k\eta_\ell W_{is}W_{it}W_{jq}W_{jm}W_{k\ell},$$

The analysis is summarized in Table G.7. We note that J_7 can be written as

$$J_7 = \frac{1}{v^2} \sum_{i,j,\ell(dist)} \beta_{ij\ell} W_{is} W_{jt} W_{jq} W_{\ell m} W_{\ell i}, \qquad \text{where} \quad \beta_{ij\ell} \equiv \sum_{k \notin \{i,j,\ell\}} \eta_k^2.$$

Although the values of $\beta_{ij\ell}$ change with indices, they have a common upper bound of $C\|\theta\|^2$. We treat $\beta_{ij\ell}$ as $\|\theta\|^2$ in Table G.7, as this doesn't change the bounds but simplifies notations. Recall that the definition of J_5-J_8 contains a factor of $\frac{1}{v^2}$ in front of the sum, where $v \asymp \|\theta\|_1^2$ by (80). Hence, to get a desired bound, we only need that each row in the third column of Table G.6 is

$$o(\|\theta\|^8 \|\theta\|_1^8).$$

This is true. We thus conclude that

$$\max\left\{\mathbb{E}[J_5^2],\, \mathbb{E}[J_6^2],\, \mathbb{E}[J_7^2],\, \mathbb{E}[J_8^2]\right\} = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider J_9-J_{10}. They can be analyzed in the same way as we did for J_1-J_8. To save space, we only give a simplified proof for the case of $\|\theta\| \gg \alpha[\log(n)]^{5/2}$. For $1 \ll \|\theta\| \leq C\alpha[\log(n)]^{5/2}$, the proof is similar to those in Tables G.6-G.7, which is omitted. For a constant $C_0 > 0$ to be decided, we introduce an event

$$E = \cap_{i=1}^n E_i, \qquad \text{where} \quad E_i = \left\{\sqrt{v}|G_i| \leq C_0\sqrt{\theta_i \|\theta\|_1 \log(n)}\right\}. \tag{271}$$

Recall that $\sqrt{v}G_i = \sqrt{v}(\tilde{\eta}_i - \eta_i) = \sum_{j \neq i}(A_{ij} - \mathbb{E}A_{ij})$. The variables $\{A_{ij}\}_{j \neq i}$ are mutually independent, satisfying that $|A_{ij} - \mathbb{E}A_{ij}| \leq 1$ and $\sum_j \mathrm{Var}(A_{ij}) \leq \sum_j \theta_i\theta_j \leq \theta_i\|\theta\|_1$. By Bernstein's inequality, for large n, the probability of E_i^c is $O(n^{-C_0/4.1})$. Applying the probability union bound, we find that the probability of E^c is $O(n^{-C_0/2.01})$. Recall that $V = \sum_{i,j:i\neq j} A_{ij}$. On the event E^c, if $V = 0$ (i.e., the network has no edges), then $\widetilde{Q}_n^* = Q_n^* = 0$; otherwise, $V \geq 1$ and $|\widetilde{Q}_n^* - Q_n^*| \leq n^4$. Combining these results gives

$$\mathbb{E}\big[|\widetilde{Q}_n^* - Q_n^*|^2 \cdot I_{E^c}\big] \leq n^4 \cdot O(n^{-C_0/2.01}).$$

With an properly large C_0, the right hand side is $o(\|\theta\|^8)$. Hence, it suffices to focus on the event E. On the event E,

$$\begin{aligned}
|J_9| &\leq \sum_{i,j,k,\ell} |\eta_k \widetilde{\Omega}_{\ell i}||G_i G_j^2 G_k G_\ell| \\
&\leq C \sum_{i,j,k,\ell} (\alpha\theta_i\theta_k\theta_\ell) \frac{\sqrt{\theta_i\theta_j^2\theta_k\theta_\ell \|\theta\|_1^5 [\log(n)]^5}}{\sqrt{v^5}} \\
&\leq \frac{C\alpha[\log(n)]^{5/2}}{\sqrt{\|\theta\|_1^5}} \Big(\sum_i \theta_i^{3/2}\Big)\Big(\sum_j \theta_j\Big)\Big(\sum_k \theta_k^{3/2}\Big)\Big(\sum_\ell \theta_\ell^{3/2}\Big) \\
&\leq \frac{C\alpha[\log(n)]^{5/2}}{\sqrt{\|\theta\|_1^3}} \Big(\sum_i \theta_i^{3/2}\Big)^3 \\
&\leq \frac{C\alpha[\log(n)]^{5/2}}{\sqrt{\|\theta\|_1^3}} \Big(\sum_i \theta_i^2\Big)^{3/2}\Big(\sum_i \theta_i\Big)^{3/2} \\
&\leq C\alpha[\log(n)]^{5/2}\|\theta\|^3,
\end{aligned}$$

TABLE G.7

Analysis of J_5-J_8. In the second column, the variables in brackets are paired W terms.

	Types of summand	Terms in mean-squared	Bound
J_5	$\eta_j\eta_k W_{\ell i}^3 W_{jk}^2$	$\eta_j\eta_k\eta_{j'}\eta_{k'}\,\mathbb{E}[W_{\ell i}^3 W_{jk}^2 W_{\ell' i'}^3 W_{j'k'}^2] \leq C\theta_i\theta_j^2\theta_k^2\theta_\ell\theta_{i'}\theta_{j'}^2\theta_{k'}^2\theta_{\ell'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_1^4$
	$\eta_j\eta_k W_{\ell i}^3(W_{jt}W_{kq})$	$\eta_j^2\eta_k^2\,\mathbb{E}[(W_{jt}^2 W_{kq}^2)W_{\ell i}^3 W_{\ell' i'}^3] \leq C\theta_i\theta_j^3\theta_k^3\theta_\ell\theta_t\theta_q\theta_{i'}\theta_{\ell'}$	$\lVert\theta\rVert_3^6\lVert\theta\rVert_1^6$
	$\eta_j\eta_k(W_{\ell i}^2 W_{is})W_{jk}^2$	$\eta_j\eta_k\eta_{j'}\eta_{k'}\,\mathbb{E}[(W_{is}^2)W_{\ell i}^2 W_{jk}^2 W_{\ell' i}^2 W_{j'k'}^2] \leq C\theta_i^3\theta_j^2\theta_k^2\theta_\ell\theta_s\theta_{j'}^2\theta_{k'}^2\theta_{\ell'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\eta_j\eta_k(W_{\ell i}^2 W_{is})(W_{jt}W_{kq})$	$\eta_j^2\eta_k^2\,\mathbb{E}[(W_{is}^2)W_{jt}^2 W_{kq}^2)W_{\ell i}^2 W_{\ell' i}^2] \leq C\theta_i^3\theta_j^3\theta_k^3\theta_\ell\theta_s\theta_t\theta_q\theta_{\ell'}$	$\lVert\theta\rVert_3^9\lVert\theta\rVert_1^5$
	$\eta_j\eta_k W_{\ell i}^2 W_{ij}^2 W_{kq}$	$\eta_j\eta_k^2\eta_{j'}\,\mathbb{E}[(W_{kq}^2)W_{\ell i}^2 W_{ij}^2 W_{\ell' i'}^2 W_{i'j'}^2] \leq C\theta_i^2\theta_j^2\theta_k^3\theta_\ell\theta_q\theta_{i'}^2\theta_{j'}^2\theta_{\ell'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\eta_j\eta_k(W_{\ell i}W_{is}W_{\ell m})W_{jk}^2$	$\eta_j\eta_k\eta_{j'}\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell m}^2)W_{jk}^2 W_{j'k'}^2] \leq C\theta_i^2\theta_j^2\theta_k^2\theta_\ell^2\theta_s\theta_m\theta_{j'}^2\theta_{k'}^2$	$\lVert\theta\rVert^{12}\lVert\theta\rVert_1^2$
	$\eta_j\eta_k(W_{\ell i}W_{is}W_{\ell m})(W_{jt}W_{kq})$	$\eta_j^2\eta_k^2\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell m}^2 W_{jt}^2 W_{kq}^2)] \leq C\theta_i^2\theta_j^3\theta_k^3\theta_\ell^2\theta_s\theta_t\theta_q\theta_m$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^4$
	$\eta_j\eta_k W_{\ell i}W_{ij}^2 W_{\ell m}W_{kq}$	$\eta_j\eta_k^2\eta_{j'}\,\mathbb{E}[(W_{\ell i}^2 W_{\ell m}^2 W_{kq}^2)W_{ij}^2 W_{ij'}^2] \leq C\theta_i^3\theta_j^2\theta_k^3\theta_\ell^2\theta_q\theta_m\theta_{j'}^2$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^6\lVert\theta\rVert_1^2$
J_6	$\eta_k\eta_\ell W_{\ell i}^2 W_{jt}^2 W_{km}$	$\eta_k^2\eta_\ell\eta_{\ell'}\,\mathbb{E}[(W_{km}^2)W_{\ell i}^2 W_{jt}^2 W_{\ell' i'}^2 W_{j't'}^2] \leq C\theta_i\theta_j\theta_k^3\theta_\ell^2\theta_t\theta_m\theta_{i'}\theta_{j'}\theta_{\ell'}^2\theta_{t'}$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^3\lVert\theta\rVert_1^7$
	$\eta_k\eta_\ell W_{\ell i}^2 W_{jk}^3$	$\eta_k\eta_\ell\eta_{k'}\eta_{\ell'}\,\mathbb{E}[W_{\ell i}^2 W_{jk}^3 W_{\ell' i'}^2 W_{j'k'}^3] \leq C\theta_i\theta_j\theta_k^2\theta_\ell^2\theta_{i'}\theta_{j'}\theta_{k'}^2\theta_{\ell'}^2$	$\lVert\theta\rVert^8\lVert\theta\rVert_1^4$
	$\eta_k\eta_\ell W_{\ell i}^2(W_{jt}W_{jq})W_{km}$	$\eta_k^2\eta_\ell\eta_{\ell'}\,\mathbb{E}[(W_{jt}^2 W_{jq}^2 W_{km}^2)W_{\ell i}^2 W_{\ell' i'}^2] \leq C\theta_i\theta_j^2\theta_k^3\theta_\ell^2\theta_t\theta_q\theta_m\theta_{i'}\theta_{\ell'}^2$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^3\lVert\theta\rVert_1^5$
	$\eta_k\eta_\ell W_{\ell i}^2 W_{jk}^2 W_{jq}$	$\eta_k\eta_\ell\eta_{k'}\eta_{\ell'}\,\mathbb{E}[(W_{jq}^2)W_{\ell i}^2 W_{jk}^2 W_{\ell' i'}^2 W_{jk'}^2] \leq C\theta_i\theta_j^3\theta_k^2\theta_\ell^2\theta_q\theta_{i'}\theta_{k'}^2\theta_{\ell'}^2$	$\lVert\theta\rVert^8\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\eta_k\eta_\ell(W_{\ell i}W_{is})W_{jt}^2 W_{km}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{km}^2)W_{jt}^2 W_{j't'}^2] \leq C\theta_i^2\theta_j\theta_k^3\theta_\ell^3\theta_s\theta_t\theta_m\theta_{j'}\theta_{t'}$	$\lVert\theta\rVert^2\lVert\theta\rVert_3^6\lVert\theta\rVert_1^6$
	$\eta_k\eta_\ell W_{\ell i}W_{ij}^3 W_{km}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{\ell i}^2 W_{km}^2)W_{ij}^3 W_{ij'}^3] \leq C\theta_i^3\theta_j\theta_k^3\theta_\ell^3\theta_m\theta_{j'}$	$\lVert\theta\rVert_3^9\lVert\theta\rVert_1^3$
	$\eta_k\eta_\ell W_{\ell i}W_{is}W_{jk}^3$	$\eta_k\eta_\ell^2\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2)W_{jk}^3 W_{j'k'}^3] \leq C\theta_i^2\theta_j\theta_k^2\theta_\ell^3\theta_s\theta_{j'}\theta_{k'}^2$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\eta_k\eta_\ell W_{\ell i}W_{ik}^2 W_{jt}^2$	$\eta_k\eta_\ell^2\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2)W_{ik}^2 W_{jt}^2 W_{ik'}^2 W_{j't'}^2] \leq C\theta_i^3\theta_j\theta_k^2\theta_\ell^3\theta_t\theta_{j'}\theta_{k'}^2\theta_{t'}$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^4$
	$\eta_k\eta_\ell(W_{\ell i}W_{is})(W_{jt}W_{jq})W_{km}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{jt}^2 W_{jq}^2 W_{km}^2)] \leq C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^3\theta_s\theta_t\theta_q\theta_m$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^4$
	$\eta_k\eta_\ell W_{\ell i}W_{ij}^2 W_{jq}W_{km}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{\ell i}^2 W_{jq}^2 W_{km}^2)W_{ij}^4] \leq C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^3\theta_q\theta_m$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^2$
	$\eta_k\eta_\ell W_{\ell i}W_{is}W_{jk}^2 W_{jq}$	$\eta_k\eta_\ell^2\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{jq}^2)W_{jk}^2 W_{jk'}^2] \leq C\theta_i^2\theta_j^3\theta_k^2\theta_\ell^3\theta_s\theta_q\theta_{k'}^2$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^6\lVert\theta\rVert_1^2$
	$\eta_k\eta_\ell W_{\ell i}W_{ik}^2 W_{jt}W_{jq}$	$\eta_k\eta_\ell^2\eta_{k'}\,\mathbb{E}[(W_{\ell i}^2 W_{jt}^2 W_{jq}^2)W_{ik}^2 W_{ik'}^2] \leq C\theta_i^3\theta_j^2\theta_k^2\theta_\ell^3\theta_t\theta_q\theta_{k'}^2$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^6\lVert\theta\rVert_1^2$
J_7	$\lVert\theta\rVert^2 W_{\ell i}^3 W_{jt}^2$	$\lVert\theta\rVert^4\,\mathbb{E}[W_{\ell i}^3 W_{jt}^2 W_{\ell' i'}^3 W_{j't'}^2] \leq C\lVert\theta\rVert^4\theta_i\theta_j\theta_\ell\theta_t\theta_{i'}\theta_{j'}\theta_{\ell'}\theta_{t'}$	$\lVert\theta\rVert^4\lVert\theta\rVert_1^8$
	$\lVert\theta\rVert^2 W_{\ell i}^3(W_{jt}W_{jq})$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{jt}^2 W_{jq}^2)W_{\ell i}^3 W_{\ell' i'}^3] \leq C\lVert\theta\rVert^4\theta_i\theta_j^2\theta_\ell\theta_t\theta_q\theta_{i'}\theta_{\ell'}$	$\lVert\theta\rVert^6\lVert\theta\rVert_1^6$
	$\lVert\theta\rVert^2(W_{\ell i}^2 W_{is})W_{jt}^2$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{is}^2)W_{\ell i}^2 W_{jt}^2 W_{\ell' i}^2 W_{j't'}^2] \leq C\lVert\theta\rVert^4\theta_i^3\theta_j\theta_\ell\theta_s\theta_t\theta_{j'}\theta_{\ell'}\theta_{t'}$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^3\lVert\theta\rVert_1^7$
	$\lVert\theta\rVert^2 W_{\ell i}^2 W_{ij}^3$	$\lVert\theta\rVert^4\,\mathbb{E}[W_{\ell i}^2 W_{ij}^3 W_{\ell' i'}^2 W_{i'j'}^3] \leq C\lVert\theta\rVert^4\theta_i^2\theta_j\theta_\ell\theta_{i'}^2\theta_{j'}\theta_{\ell'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_1^4$
	$\lVert\theta\rVert^2(W_{\ell i}^2 W_{is})(W_{jt}W_{jq})$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{is}^2 W_{jt}^2 W_{jq}^2)W_{\ell i}^2 W_{\ell' i}^2] \leq C\lVert\theta\rVert^4\theta_i^3\theta_j^2\theta_\ell\theta_s\theta_t\theta_q\theta_{\ell'}$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^3\lVert\theta\rVert_1^5$
	$\lVert\theta\rVert^2 W_{\ell i}^2 W_{ij}^2 W_{jq}$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{jq}^2)W_{\ell i}^2 W_{ij}^2 W_{\ell' i'}^2 W_{i'j}^2] \leq C\lVert\theta\rVert^4\theta_i^2\theta_j^3\theta_\ell\theta_q\theta_{i'}^2\theta_{\ell'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\lVert\theta\rVert^2(W_{\ell i}W_{is}W_{\ell m})W_{jt}^2$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell m}^2)W_{jt}^2 W_{j't'}^2] \leq C\lVert\theta\rVert^4\theta_i^2\theta_j\theta_\ell^2\theta_s\theta_t\theta_m\theta_{j'}\theta_{t'}$	$\lVert\theta\rVert^8\lVert\theta\rVert_1^6$
	$\lVert\theta\rVert^2 W_{\ell i}W_{ij}^3 W_{\ell m}$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{\ell i}^2 W_{\ell m}^2)W_{ij}^3 W_{ij'}^3] \leq C\lVert\theta\rVert^4\theta_i^3\theta_j\theta_\ell^2\theta_m\theta_{j'}$	$\lVert\theta\rVert^6\lVert\theta\rVert_3^3\lVert\theta\rVert_1^3$
	$\lVert\theta\rVert^2(W_{\ell i}W_{is}W_{\ell m})(W_{jt}W_{jq})$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{\ell i}^2 W_{is}^2 W_{\ell m}^2 W_{jt}^2 W_{jq}^2)] \leq C\lVert\theta\rVert^4\theta_i^2\theta_j^2\theta_\ell^2\theta_s\theta_t\theta_q\theta_m$	$\lVert\theta\rVert^{10}\lVert\theta\rVert_1^4$
	$\lVert\theta\rVert^2 W_{\ell i}W_{ij}^2 W_{\ell m}W_{jq}$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{\ell i}^2 W_{\ell m}^2 W_{jq}^2)W_{ij}^4] \leq C\lVert\theta\rVert^4\theta_i^2\theta_j^2\theta_\ell^2\theta_q\theta_m$	$\lVert\theta\rVert^{10}\lVert\theta\rVert_1^2$
	$\lVert\theta\rVert^2 W_{\ell i}W_{ij}^2 W_{\ell j}^2$	$\lVert\theta\rVert^4\,\mathbb{E}[(W_{\ell i}^2)W_{ij}^2 W_{\ell j}^2 W_{ij'}^2 W_{\ell j'}^2] \leq C\lVert\theta\rVert^4\theta_i^3\theta_j^2\theta_\ell^3\theta_{j'}^2$	$\lVert\theta\rVert^8\lVert\theta\rVert_3^6$
J_8	$\eta_k\eta_\ell W_{ij}^4 W_{k\ell}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{k\ell}^2)W_{ij}^4 W_{i'j'}^4] \leq C\theta_i\theta_j\theta_k^3\theta_\ell^3\theta_{i'}\theta_{j'}$	$\lVert\theta\rVert_3^6\lVert\theta\rVert_1^4$
	$\eta_k\eta_\ell(W_{ij}^3 W_{is})W_{k\ell}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{is}^2 W_{k\ell}^2)W_{ij}^3 W_{ij'}^3] \leq C\theta_i^3\theta_j\theta_k^3\theta_\ell^3\theta_s\theta_{j'}$	$\lVert\theta\rVert_3^9\lVert\theta\rVert_1^3$
	$\eta_k\eta_\ell(W_{ij}^2 W_{is}W_{jq})W_{k\ell}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{is}^2 W_{jq}^2 W_{k\ell}^2)W_{ij}^4] \leq C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^3\theta_s\theta_q$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^2$
	$\eta_k\eta_\ell(W_{is}W_{it}W_{jq}W_{jm})W_{k\ell}$	$\eta_k^2\eta_\ell^2\,\mathbb{E}[(W_{is}^2 W_{it}^2 W_{jq}^2 W_{jm}^2 W_{k\ell}^2)] \leq C\theta_i^2\theta_j^2\theta_k^3\theta_\ell^3\theta_s\theta_t\theta_q\theta_m$	$\lVert\theta\rVert^4\lVert\theta\rVert_3^6\lVert\theta\rVert_1^4$
	$\eta_k\eta_\ell W_{is}^2 W_{jq}W_{jm}W_{k\ell}$	$\eta_k^2\eta_\ell^2\mathbb{E}[(W_{jq}^2 W_{jm}^2 W_{k\ell}^2)W_{is}^2 W_{i's'}^2] \leq C\theta_i\theta_j^2\theta_k^3\theta_\ell^3\theta_s\theta_q\theta_m\theta_{i'}\theta_{s'}$	$\lVert\theta\rVert^2\lVert\theta\rVert_3^6\lVert\theta\rVert_1^6$
	$\eta_k\eta_\ell W_{is}^2 W_{jq}^2 W_{k\ell}$	$\eta_k^2\eta_\ell^2\mathbb{E}[(W_{k\ell}^2)W_{is}^2 W_{jq}^2 W_{i's'}^2 W_{j'q'}^2] \leq C\theta_i\theta_j\theta_k^3\theta_\ell^3\theta_s\theta_q\theta_{i'}\theta_{j'}\theta_{s'}\theta_{q'}$	$\lVert\theta\rVert_3^6\lVert\theta\rVert_1^8$

where the second last line is from the Cauchy-Schwarz inequality. Since $\|\theta\| \gg \alpha[\log(n)]^{5/2}$, the right hand side is $o(\|\theta\|^4)$, which implies that $|J_9|^2 = o(\|\theta\|^8)$. Similarly, on the event E,

$$|J_{10}| \leq \sum_{i,j,k,\ell} |\eta_\ell \widetilde{\Omega}_{\ell i}||G_i G_j^2 G_k^2|$$

$$\leq C \sum_{i,j,k,\ell} (\alpha\theta_i\theta_\ell^2) \frac{\sqrt{\theta_i\theta_j^2\theta_k^2\|\theta\|_1^5[\log(n)]^5}}{\sqrt{v^5}}$$

$$\le \frac{C\alpha[\log(n)]^{5/2}}{\sqrt{\|\theta\|_1^5}} \Big(\sum_i \theta_i^{3/2}\Big)\Big(\sum_j \theta_j\Big)\Big(\sum_k \theta_k\Big)\Big(\sum_\ell \theta_\ell^2\Big)$$
$$\le \frac{C\alpha[\log(n)]^{5/2}}{\sqrt{\|\theta\|_1^5}} (\|\theta\|\sqrt{\|\theta\|_1})\|\theta\|_1^2\|\theta\|^2$$
$$\le C\alpha[\log(n)]^{5/2}\|\theta\|^3;$$

again, the right hand side is $o(\|\theta\|^4)$. Combining the above gives

$$\max\big\{\mathbb{E}[J_9^2],\, \mathbb{E}[J_{10}^2]\big\} = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

So far, we have proved: for each R_k with $N_W^* = 5$, it satisfies $\mathbb{E}[R_k^2] = o(\|\theta\|^8)$. This is sufficient to guarantee (266)-(267) for $X = R_k$.

G.4.10.3. Analysis of post-expansion sums with $N_W^* = 6$. There are 7 such terms, including R_{19}-R_{20}, R_{23}-R_{24}, R_{29}-R_{30}, and R_{32}. We plug in the definition of $\tilde{r}_{ij}$ and δ_{ij} and neglect all factors of $\frac{v}{V}$ (see the explanation in (266)-(267)). It gives ($G_i = \tilde{\eta}_i - \eta_i$):

$$R_{19} = \sum_{i,j,k,\ell(dist)} G_i G_j^2 G_k W_{k\ell} W_{\ell i},$$

$$R_{20} = \sum_{i,j,k,\ell(dist)} G_i G_j W_{jk} G_k G_\ell W_{\ell i},$$

$$R_{23} = \sum_{i,j,k,\ell(dist)} G_i G_j^2 G_k(\eta_k G_\ell^2 \eta_i + 2G_k \eta_\ell G_\ell \eta_i + G_k \eta_\ell^2 G_i)$$
$$= \sum_{i,j,k,\ell(dist)} \eta_i \eta_k G_i G_j^2 G_k G_\ell^2 + 2\sum_{i,j,k,\ell(dist)} \eta_i \eta_\ell G_i G_j^2 G_k^2 G_\ell + \sum_{i,j,k,\ell(dist)} \eta_\ell^2 G_i^2 G_j^2 G_k^2$$
$$= 3\sum_{i,j,k,\ell(dist)} \eta_i \eta_k G_i G_j^2 G_k G_\ell^2 + \sum_{i,j,k,\ell(dist)} \eta_\ell^2 G_i^2 G_j^2 G_k^2,$$

$$R_{24} = \sum_{i,j,k,\ell(dist)} G_i G_j(\eta_j G_k + G_j \eta_k) G_k G_\ell(\eta_\ell G_i + G_\ell \eta_i)$$
$$= 4\sum_{i,j,k,\ell(dist)} \eta_j \eta_\ell G_i^2 G_j G_k^2 G_\ell,$$

$$R_{29} = \sum_{i,j,k,\ell(dist)} G_i G_j^2 G_k(\eta_k G_\ell + G_k \eta_\ell) W_{\ell i}$$
$$= \sum_{i,j,k,\ell(dist)} \eta_k G_i G_j^2 G_k G_\ell W_{\ell i} + \sum_{i,j,k,\ell(dist)} \eta_\ell G_i G_j^2 G_k^2 W_{\ell i},$$

$$R_{30} = 2\sum_{i,j,k,\ell(dist)} G_i G_j(\eta_j G_k) G_k G_\ell W_{\ell i} = 2\sum_{i,j,k,\ell(dist)} \eta_j G_i G_j G_k^2 G_\ell W_{\ell i},$$

$$R_{32} = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{\ell i} G_i G_j^2 G_k^2 G_\ell.$$

Each expression above belongs to one of the following types:

$$K_1 = \sum_{i,j,k,\ell(dist)} G_i G_j^2 G_k W_{k\ell} W_{\ell i}, \qquad K_2 = \sum_{i,j,k,\ell(dist)} G_i G_j G_k G_\ell W_{jk} W_{\ell i},$$

$$K_3 = \sum_{i,j,k,\ell(dist)} \eta_k G_i G_j^2 G_k G_\ell W_{\ell i}, \qquad K_4 = \sum_{i,j,k,\ell(dist)} \eta_\ell G_i G_j^2 G_k^2 W_{\ell i},$$

$$K_5 = \sum_{i,j,k,\ell(dist)} \eta_i \eta_k G_i G_j^2 G_k G_\ell^2, \qquad K_5' = \sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ik} G_i G_j^2 G_k G_\ell^2,$$

$$K_6 = \sum_{i,j,k,\ell(dist)} \eta_\ell^2 G_i^2 G_j^2 G_k^2.$$

Since $|\eta_i\eta_k| \leq C\theta_i\theta_k$ and $|\widetilde{\Omega}_{ik}| \leq C\alpha\theta_i\theta_k$, the study of K_5 and K_5' are similar; we thus omit the analysis of K_5'. We now study K_1-K_6.

Consider K_1. Re-write

$$K_1 = \frac{1}{v^2} \sum_{\substack{i,j,k,\ell(dist)\\ s\neq i, t\neq j, q\neq j, m\neq k}} W_{is}W_{jt}W_{jq}W_{km}W_{k\ell}W_{\ell i}.$$

Note that $W_{km}W_{k\ell}W_{\ell i}W_{is}$ has four different cases: (a) $W_{k\ell}^2 W_{\ell i}^2$, (b) $W_{k\ell}^2 W_{\ell i}W_{is}$, (c) $W_{k\ell}W_{\ell i}W_{ik}^2$, and (d) $W_{k\ell}W_{\ell i}W_{km}W_{is}$. At the same time, $W_{jt}W_{jq}$ has two cases: (i) W_{jk}^2 and (ii) $W_{jt}W_{jq}$. This gives at least $4 \times 2 = 8$ cases. Each case may have sub-cases, e.g., for $(W_{k\ell}^2 W_{\ell i}W_{is})W_{jt}^2$, if $(s,t) = (j,i)$, it becomes $W_{k\ell}^2 W_{\ell i} W_{ij}^3$. By direct calculations, all possible cases of the summand are as follows:

$$
\begin{aligned}
&(W_{k\ell}^2 W_{\ell i}^2)W_{jt}^2, \quad (W_{k\ell}^2 W_{\ell i}^2)(W_{jt}W_{jq}), \quad (W_{k\ell}^2 W_{\ell i}W_{is})W_{jt}^2,\\
&W_{k\ell}^2 W_{\ell i}W_{ij}^3, \quad (W_{k\ell}^2 W_{\ell i}W_{is})(W_{jt}W_{jq}), \quad W_{k\ell}^2 W_{\ell i}W_{ij}^2 W_{jq},\\
&(W_{k\ell}W_{\ell i}W_{ik}^2)W_{jt}^2, \quad (W_{k\ell}W_{\ell i}W_{ik}^2)(W_{jt}W_{jq}),\\
&(W_{k\ell}W_{\ell i}W_{km}W_{is})W_{jt}^2, \quad W_{k\ell}W_{\ell i}W_{km}W_{ij}^3,\\
&(W_{k\ell}W_{\ell i}W_{km}W_{is})(W_{jt}W_{jq}), \quad W_{k\ell}W_{\ell i}W_{km}W_{ij}^2 W_{jq},\\
&W_{k\ell}W_{\ell i}W_{kj}^2 W_{ij}^2. \qquad (272)
\end{aligned}
$$

Take the second type for example. We aim to bound $\mathbb{E}[(\sum_{i,j,k,\ell,t,q} W_{k\ell}^2 W_{\ell i}^2 W_{jt}W_{jq})^2]$, which is equal to

$$\sum_{\substack{i,j,k,\ell,t,q\\ i',j',k',\ell',t',q'}} \mathbb{E}[W_{k\ell}^2 W_{\ell i}^2 W_{jt}W_{jq}W_{k'\ell'}^2 W_{\ell' i'}^2 W_{j't'}W_{j'q'}].$$

For the expectation to be nonzero, each single W term has to be paired with another term. The main contribution comes from the case that $W_{j't'}W_{j'q'} = W_{jt}W_{jq}$. It implies $(j',t',q') = (j,t,q)$ or $(j',t',q') = (j,q,t)$. Then, the expression above becomes

$$
\begin{aligned}
\sum_{\substack{i,j,k,\ell,t,q\\ i',k',\ell'}} \mathbb{E}[(W_{jt}^2 W_{jq}^2)W_{k\ell}^2 W_{\ell i}^2 W_{k'\ell'}^2 W_{\ell' i'}^2] &\leq C \sum_{\substack{i,j,k,\ell,t,q\\ i',k',\ell'}} \theta_i\theta_j^2\theta_k\theta_\ell^2\theta_t\theta_q\theta_{i'}\theta_{k'}\theta_{\ell'}^2\\
&\leq C\|\theta\|^6\|\theta\|_1^6.
\end{aligned}
$$

There are a total of 9 indices in this sum, which are $(i,j,k,\ell,t,q,i',k',\ell')$. Similarly, for each type of summand, when we bound the expectation of the square of its sum, we count how many indices appear in the ultimate sum. This number equals to twice of the total number of indices appearing in the summand, minus the total number of indices appearing in single W terms. For the above example, all indices appearing in the summand are (i,j,k,ℓ,t,q), while indices appearing in single W terms are (j,t,q); so, the aforementioned number is $2\times 6-3=9$. If this number if m_0, then the expectation of the square of sum of this type is bounded by $C\|\theta\|_1^{m_0}$. We note that K_1 has a factor $\frac{1}{v^2}$ in front of the sum, which brings in a factor of $\frac{C}{\|\theta\|_1^8}$ in the bound. Therefore, for any type of summand with $m_0\leq 8$, the expectation of the square of its sum is $O(1)$, which is $o(\|\theta\|^8)$. As a result, among the types in (272), we only need to consider those with $m_0\geq 9$. We are left with

$$(W_{k\ell}^2W_{\ell i}^2)W_{jt}^2, \qquad (W_{k\ell}^2W_{\ell i}^2)(W_{jt}W_{jq}), \qquad (W_{k\ell}^2W_{\ell i}W_{is})W_{jt}^2.$$

We have proved that the expectation of the square of sum of the second type of summands is bounded by $C\|\theta\|^2\|\theta\|_1^6=o(\|\theta\|^8\|\theta\|_1^8)$. For the other two types, by direct calculations,

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ t\neq j}} W_{k\ell}^2W_{\ell i}^2W_{jt}^2\Big)^2\Big] &\leq \sum_{\substack{i,j,k,\ell,t\\ i',j',k',\ell',t'}} \mathbb{E}[W_{k\ell}^2W_{\ell i}^2W_{jt}^2W_{k'\ell'}^2W_{\ell' i'}^2W_{j't'}^2]\\
&\leq \sum_{\substack{i,j,k,\ell,t\\ i',j',k',\ell',t'}} \theta_i\theta_j\theta_k\theta_\ell^2\theta_t\theta_{i'}\theta_{j'}\theta_{k'}\theta_{\ell'}^2\theta_{t'}\\
&\leq C\|\theta\|^4\|\theta\|_1^8=o(\|\theta\|^8\|\theta\|_1^8),\\
\mathbb{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ s\notin\{i,\ell\},t\neq j,\\ (s,t)\neq(j,i)}} W_{k\ell}^2W_{\ell i}W_{is}W_{jt}^2\Big)^2\Big] &\leq \sum_{\substack{i,j,k,\ell,s,t\\ j',k',t'}} \mathbb{E}[(W_{\ell i}^2W_{is}^2)W_{k\ell}^2W_{jt}^2W_{k'\ell}^2W_{j't'}^2]\\
&\leq C\sum_{\substack{i,j,k,\ell,s,t\\ j',k',t'}} \theta_i^2\theta_j\theta_k\theta_\ell^3\theta_s\theta_t\theta_{j'}\theta_{k'}\theta_{t'}\\
&\leq C\|\theta\|^2\|\theta\|_3^3\|\theta\|_1^7=o(\|\theta\|^8\|\theta\|_1^8).
\end{aligned}$$

Combining the above gives

$$\mathbb{E}[K_1^2]=o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider K_2. Re-write

$$K_2=\frac{1}{v^2}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j,q\neq k,m\neq\ell}} W_{is}W_{jt}W_{kq}W_{\ell m}W_{jk}W_{\ell i}.$$

Note that $W_{qk}W_{kj}W_{jt}$ has three cases: (a) W_{kj}^3, (b) $W_{kj}^2W_{jt}$ (or $W_{qk}W_{kj}^2$), and (c) $W_{qk}W_{kj}W_{jt}$. Simiarly, $W_{m\ell}W_{\ell i}W_{is}$ has three cases: (a) $W_{\ell i}^3$, (b) $W_{\ell i}^2W_{is}$ (or $W_{m\ell}W_{\ell i}^2$), and (c) $W_{m\ell}W_{\ell i}W_{is}$. By index symmetry, this gives $3+2+1=6$ different cases. Some case may have sub-cases, due to that (s,t) may equal to (j,i), say. By direct calculations, all possible cases of the summand are as follows:

$$\begin{aligned}
&W_{kj}^3W_{\ell i}^3, \quad W_{kj}^3(W_{\ell i}^2W_{is}), \quad W_{kj}^3(W_{m\ell}W_{\ell i}W_{is}), \quad (W_{kj}^2W_{jt})(W_{\ell i}^2W_{is}),\\
&W_{kj}^2W_{ji}^2W_{\ell i}^2, \quad (W_{kj}^2W_{jt})(W_{m\ell}W_{\ell i}W_{is}), \quad W_{kj}^2W_{ji}^2W_{m\ell}W_{\ell i},\\
&(W_{qk}W_{kj}W_{jt})(W_{m\ell}W_{\ell i}W_{is}), \quad W_{qk}W_{kj}W_{ji}^2W_{m\ell}W_{\ell i}, \quad W_{kj}W_{ji}^2W_{k\ell}^2W_{\ell i}.
\end{aligned}$$

As in the analysis of (272), we count the effective number of indices, m_0, which equals to twice of the total number of indices appearing in the summand minus the total number of indices appearing in all single-W terms. For the above types of summand, m_0 equals to

$8, 8, 8, 8, 8, 8, 7, 8, 6, 4$, respectively. None is larger than 8. We conclude that the expectation of the square of sum of each type of summand is bounded by $C\|\theta\|_1^8$. We immediately have

$$\mathbb{E}[K_2^2] = \frac{1}{v^4} \cdot C\|\theta\|_1^8 = O(1) = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider K_3. Re-write

$$K_3 = \frac{1}{v^2\sqrt{v}} \sum_{\substack{i,j,k,\ell (dist)\\ s\neq i, t\neq j, q\neq j, m\neq k, p\neq \ell}} \eta_k W_{is} W_{jt} W_{jq} W_{km} W_{\ell p} W_{\ell i}$$

Note that $W_{jt}W_{jq}W_{km}$ has four cases: (a) W_{jk}^3, (b) $W_{jk}^2 W_{jt}$ (or $W_{jk}^2 W_{jq}$), (c) $W_{jt}^2 W_{km}$, and (d) $W_{jt}W_{jq}W_{km}$. At the same time, $W_{is}W_{\ell p}W_{\ell i}$ has three cases: (a) $W_{\ell i}^3$, (b) $W_{\ell i}^2 W_{is}$ (or $W_{\ell i}^2 W_{\ell p}$), and (c) $W_{\ell i}W_{is}W_{\ell p}$. This gives $4 \times 3 = 12$ different cases. Each case may have sub-cases. For example, in the case of $\eta_k(W_{jk}^2 W_{jt})(W_{\ell i}^2 W_{is})$, if $(s,t) = (j,i)$, it becomes $\eta_k W_{jk}^2 W_{ji}^2 W_{\ell i}^2$. By direct calculations, we obtain all possible cases of summands as follows:

$$\begin{gathered}
\eta_k W_{jk}^3 W_{\ell i}^3, \quad \eta_k W_{jk}^3 (W_{\ell i}^2 W_{is}), \quad \eta_k W_{jk}^3 (W_{\ell i} W_{is} W_{\ell p}), \quad \eta_k (W_{jk}^2 W_{jt}) W_{\ell i}^3, \\
\eta_k (W_{jk}^2 W_{jt})(W_{\ell i}^2 W_{is}), \quad \eta_k W_{jk}^2 W_{ji}^2 W_{\ell i}^2, \quad \eta_k (W_{jk}^2 W_{jt})(W_{\ell i} W_{is} W_{\ell p}), \\
\eta_k W_{jk}^2 W_{ji}^2 W_{\ell i} W_{\ell p}, \quad \eta_k (W_{jt}^2 W_{km}) W_{\ell i}^3, \quad \eta_k (W_{jt}^2 W_{km})(W_{\ell i}^2 W_{is}), \quad \eta_k W_{jt}^2 W_{ki}^2 W_{\ell i}^2, \\
\eta_k (W_{jt}^2 W_{km})(W_{\ell i} W_{is} W_{\ell p}), \quad \eta_k W_{jt}^2 W_{ki}^2 W_{\ell i} W_{\ell p}, \quad \eta_k (W_{jt} W_{jq} W_{km}) W_{\ell i}^3, \\
\eta_k (W_{jt} W_{jq} W_{km})(W_{\ell i}^2 W_{is}), \quad \eta_k W_{jt} W_{ji}^2 W_{km} W_{\ell i}^2, \quad \eta_k W_{jt} W_{jq} W_{ki}^2 W_{\ell i}^2, \\
\eta_k (W_{jt} W_{jq} W_{km})(W_{\ell i} W_{is} W_{\ell p}), \quad \eta_k W_{jt} W_{ji}^2 W_{km} W_{\ell i} W_{\ell p}, \quad \eta_k W_{jt} W_{jq} W_{ki}^2 W_{\ell i} W_{\ell p}.
\end{gathered}$$

Same as before, let m_0 be the effective number of indices for each type of summand, which equals to twice of number of distinct indices appearing in the summand minus the number of distinct indices appearing in single-W terms (see (272) and text therein). By direct calculations, $m_0 \leq 10$ for all types above. It follows that, for each type of summand, the expectation of the square of their sums is bounded by

$$\frac{1}{(v\sqrt{v})^2} \cdot C\|\theta\|_1^{m_0} \leq C\|\theta\|_1^{m_0 - 10} = O(1) = o(\|\theta\|^8).$$

We immediately have

$$\mathbb{E}[K_3^2] = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider K_4. Re-write

$$K_4 = \frac{1}{v^2\sqrt{v}} \sum_{\substack{i,j,k,\ell (dist)\\ s,t,q,m,p}} \eta_\ell W_{is} W_{jt} W_{jq} W_{km} W_{kp} W_{\ell i}.$$

Note that $W_{is}W_{\ell i}$ has two cases: (a) $W_{\ell i}^2$ and (b) $W_{\ell i}W_{is}$. Moreover, there are a total of six cases for $W_{jt}W_{jq}W_{km}W_{kp}$: (a) W_{jk}^4, (b) $W_{jk}^3 W_{jt}$, (c) $W_{jk}^2 W_{jt} W_{km}$, (d) $W_{jt}^2 W_{km}^2$, (e) $W_{jt}W_{jq}W_{km}^2$, and (f) $W_{jt}W_{jq}W_{km}W_{kp}$. It gives $2 \times 6 = 12$ different cases. Each case may have some sub-cases. It turns out all different types of summand are as follows:

$$\begin{gathered}
\eta_\ell W_{\ell i}^2 W_{jk}^4, \quad \eta_\ell W_{\ell i}^2 (W_{jk}^3 W_{jt}), \quad \eta_\ell W_{\ell i}^2 (W_{jk}^2 W_{jt} W_{km}), \quad \eta_\ell W_{\ell i}^2 (W_{jt}^2 W_{km}^2), \\
\eta_\ell W_{\ell i}^2 (W_{jt} W_{jq} W_{km}^2), \quad \eta_\ell W_{\ell i}^2 (W_{jt} W_{jq} W_{km} W_{kp}), \quad \eta_\ell (W_{\ell i} W_{is}) W_{jk}^4, \\
\eta_\ell (W_{\ell i} W_{is})(W_{jk}^3 W_{jt}), \quad \eta_\ell W_{\ell i} W_{jk}^3 W_{ji}^2, \quad \eta_\ell (W_{\ell i} W_{is})(W_{jk}^2 W_{jt} W_{km}),
\end{gathered}$$

$$\eta_\ell W_{\ell i} W_{jk}^2 W_{ji}^2 W_{km}, \quad \eta_\ell (W_{\ell i} W_{is})(W_{jt}^2 W_{km}^2), \quad \eta_\ell W_{\ell i} W_{ij}^3 W_{km}^2,$$
$$\eta_\ell (W_{\ell i} W_{is})(W_{jt} W_{jq} W_{km}^2), \quad \eta_\ell W_{\ell i} W_{ij}^2 W_{jq} W_{km}^2, \quad \eta_\ell W_{\ell i} W_{jt} W_{jq} W_{ki}^3,$$
$$\eta_\ell (W_{\ell i} W_{is})(W_{jt} W_{jq} W_{km} W_{kp}), \quad \eta_\ell W_{\ell i} W_{ij}^2 W_{jq} W_{km} W_{kp}.$$

Same as before, for each type, let m_0 be the effective number of indices. It suffices to focus on cases where $m_0 \geq 11$. We are left with

$$\eta_\ell W_{\ell i}^2 (W_{jt}^2 W_{km}^2), \qquad \eta_\ell W_{\ell i}^2 (W_{jt} W_{jq} W_{km}^2), \qquad \eta_\ell (W_{\ell i} W_{is})(W_{jt}^2 W_{km}^2).$$

By direct calculations,

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ t\neq j, m\neq k}} \eta_\ell W_{\ell i}^2 W_{jt}^2 W_{km}^2\Big)\Big] &\leq \sum_{\substack{i,j,k,\ell,t,m\\ i',j',k',\ell',t',m'}} \eta_\ell \eta_{\ell'}\, \mathbb{E}[W_{\ell i}^2 W_{jt}^2 W_{km}^2 W_{\ell' i'}^2 W_{j't'}^2 W_{k'm'}^2] \\
&\leq C \sum_{\substack{i,j,k,\ell,t,m\\ i',j',k',\ell',t',m'}} \theta_i \theta_j \theta_k \theta_\ell^2 \theta_t \theta_m \theta_{i'} \theta_{j'} \theta_{k'} \theta_{\ell'}^2 \theta_{t'} \theta_{m'} \\
&\leq C\|\theta\|^4 \|\theta\|_1^{10} = o(\|\theta\|^8 \|\theta\|_1^{10}),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ t\neq j, q\neq j, m\neq k\\ t\neq q}} \eta_\ell W_{\ell i}^2 W_{jt} W_{jq} W_{km}^2\Big)\Big] &\leq \sum_{\substack{i,j,k,\ell,t,q,m\\ i',k',\ell',m'}} \eta_\ell \eta_{\ell'}\, \mathbb{E}[(W_{jt}^2 W_{jq}^2) W_{\ell i}^2 W_{km}^2 W_{\ell' i'}^2 W_{k'm'}^2] \\
&\leq C \sum_{\substack{i,j,k,\ell,t,q,m\\ i',k',\ell',m'}} \theta_i \theta_j^2 \theta_k \theta_\ell^2 \theta_t \theta_q \theta_m \theta_{i'} \theta_{k'} \theta_{\ell'}^2 \theta_{m'} \\
&\leq C\|\theta\|^6 \|\theta\|_1^8 = o(\|\theta\|^8 \|\theta\|_1^{10}),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{\substack{i,j,k,\ell(dist)\\ s\neq i, t\neq j, m\neq k\\ (s,t)\neq(j,i),(s,m)\neq(k,i)}} \eta_\ell W_{\ell i} W_{is} W_{jt}^2 W_{km}^2\Big)\Big] &\leq C \sum_{\substack{i,j,k,\ell,s,t,m\\ j',k',t',m'}} \eta_\ell^2\, \mathbb{E}[(W_{\ell i}^2 W_{is}^2) W_{jt}^2 W_{km}^2 W_{j't'}^2 W_{k'm'}^2] \\
&\leq C \sum_{\substack{i,j,k,\ell,s,t,m\\ j',k',t',m'}} \theta_i^2 \theta_j \theta_k \theta_\ell^3 \theta_s \theta_t \theta_m \theta_{j'} \theta_{k'} \theta_{t'} \theta_{m'} \\
&\leq C\|\theta\|^2 \|\theta\|_3^3 \|\theta\|_1^9 = o(\|\theta\|^8 \|\theta\|_1^{10}).
\end{aligned}$$

It follows that

$$\mathbb{E}[K_4^2] \leq \frac{1}{(v^2\sqrt{v})^2} \cdot o(\|\theta\|^8 \|\theta\|_1^{10}) = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider K_5-K_6. To save space, we only present the proof for the case of $\|\theta\| \gg [\log(n)]^{3/2}$. When $1 \ll \|\theta\| \leq C[\log(n)]^{3/2}$, we can bound $\mathbb{E}[K_5^2]$ and $\mathbb{E}[K_6^2]$ in the same way as in the study of J_1-J_8, so the proof is omitted. Let E be the event defined in (271). We have argued that it suffices to focus on the event E. On this event, $|G_i| \leq C\sqrt{\theta_i \|\theta\|_1 \log(n)/v}$. It follows that

$$|K_5| \leq C \sum_{i,j,k,\ell} (\theta_i \theta_k) \frac{\sqrt{\theta_i \theta_j^2 \theta_k \theta_\ell^2}\, \|\theta\|_1^3 [\log(n)]^3}{v^3}$$

$$
\begin{aligned}
&\le \frac{C[\log(n)]^3}{\|\theta\|_1^3}\Big(\sum_i \theta_i^{3/2}\Big)\Big(\sum_j \theta_j\Big)\Big(\sum_k \theta_k^{3/2}\Big)\Big(\sum_\ell \theta_\ell\Big)\\
&\le \frac{C[\log(n)]^3}{\|\theta\|_1^3}\big(\|\theta\|\sqrt{\|\theta\|_1}\big)^2\|\theta\|_1^2\\
&\le C[\log(n)]^3\|\theta\|^2,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality $(\sum_i \theta_i^{3/2}) \le \|\theta\|\sqrt{\|\theta\|_1}$. Similarly,

$$
\begin{aligned}
|K_6| &\le C\sum_{i,j,k,\ell}\theta_\ell^2\cdot\frac{\theta_i\theta_j\theta_k\|\theta\|_1^3[\log(n)]^3}{v^3}\\
&\le \frac{C[\log(n)]^3}{\|\theta\|_1^3}\sum_{i,j,k,\ell}\theta_i\theta_j\theta_k\theta_\ell^2\\
&\le C[\log(n)]^3\|\theta\|^2.
\end{aligned}
$$

When $\|\theta\| \gg [\log(n)]^{3/2}$, both right hand sides are $o(\|\theta\|^4)$. We immediately have

$$
\max\big\{\mathbb{E}[K_5^2],\, \mathbb{E}[K_6^2]\big\} = o(\|\theta\|^8).
$$

We have proved: Each R_k with $N_W^* = 6$ satisfies $\mathbb{E}[R_k^2] = o(\|\theta\|^8)$. This is sufficient to guarantee (266)-(267) for $X = R_k$.

G.4.10.4. Analysis of terms with $N_W^* \ge 7$. There are 3 such terms, R_{31}, R_{33} and R_{34}. Consider R_{31}. By definition,

$$
R_{31} = \sum_{i,j,k,\ell(dist)} G_iG_j^2G_k^2G_\ell W_{\ell i} = \frac{1}{v^3}\sum_{\substack{i,j,k,\ell(dist)\\ s\neq i,t\neq j,q\neq j,\\ m\neq k,p\neq k,y\neq \ell}} W_{is}W_{jt}W_{jq}W_{km}W_{kp}W_{\ell y}W_{\ell i}.
$$

We note that $W_{\ell i}W_{is}W_{\ell y}$ has three cases: (a) $W_{\ell i}^3$, (b) $W_{\ell i}^2W_{is}$, and (c) $W_{\ell i}W_{is}W_{\ell y}$. Moreover, $W_{jt}W_{jq}W_{km}W_{kp}$ has six cases: (a) W_{jk}^4, (b) $W_{jk}^3W_{jt}$, (c) $W_{jk}^2W_{jt}W_{km}$, (d) $W_{jt}^2W_{km}^2$, (e) $W_{jt}W_{jq}W_{km}^2$, and (f) $W_{jt}W_{jq}W_{km}W_{kp}$. This gives $3\times 6 = 18$ different cases. Since each case may have sub-cases, we end up with the following different types:

$$
\begin{aligned}
&W_{\ell i}^3W_{jk}^4, \quad W_{\ell i}^3(W_{jk}^3W_{jt}), \quad W_{\ell i}^3(W_{jk}^2W_{jt}W_{km}), \quad W_{\ell i}^3(W_{jt}^2W_{km}^2),\\
&W_{\ell i}^3(W_{jt}W_{jq}W_{km}^2), \quad W_{\ell i}^3(W_{jt}W_{jq}W_{km}W_{kp}), \quad (W_{\ell i}^2W_{is})W_{jk}^4,\\
&(W_{\ell i}^2W_{is})(W_{jk}^3W_{jt}), \quad W_{\ell i}^2W_{jk}^3W_{ji}^2, \qquad (W_{\ell i}^2W_{is})(W_{jk}^2W_{jt}W_{km}),\\
&W_{\ell i}^2W_{jk}^2W_{ji}^2W_{km}, \quad (W_{\ell i}^2W_{is})(W_{jt}^2W_{km}^2), \quad W_{\ell i}^2W_{ij}^3W_{km}^2,\\
&(W_{\ell i}^2W_{is})(W_{jt}W_{jq}W_{km}^2), \quad W_{\ell i}^2W_{ij}^2W_{jq}W_{km}^2, \quad W_{\ell i}^2W_{jt}W_{jq}W_{ki}^3,\\
&(W_{\ell i}^2W_{is})(W_{jt}W_{jq}W_{km}W_{kp}), \quad W_{\ell i}^2W_{ij}^2W_{jq}W_{km}W_{kp},\\
&(W_{\ell i}W_{is}W_{\ell y})W_{jk}^4, \quad (W_{\ell i}W_{is}W_{\ell y})(W_{jk}^3W_{jt}), \quad W_{\ell i}W_{\ell y}W_{jk}^3W_{ji}^2,\\
&(W_{\ell i}W_{is}W_{\ell y})(W_{jk}^2W_{jt}W_{km}), \quad W_{\ell i}W_{\ell y}W_{jk}^2W_{ji}^2W_{km}, \quad W_{\ell i}W_{jk}^2W_{ji}^2W_{k\ell}^2,\\
&(W_{\ell i}W_{is}W_{\ell y})(W_{jt}^2W_{km}^2), \quad W_{\ell i}W_{\ell y}W_{ji}^3W_{km}^2, \quad W_{\ell i}W_{ji}^3W_{k\ell}^3,\\
&(W_{\ell i}W_{is}W_{\ell y})(W_{jt}W_{jq}W_{km}^2), \quad W_{\ell i}W_{\ell y}W_{ji}^2W_{jq}W_{km}^2, \quad W_{\ell i}W_{\ell y}W_{jt}W_{jq}W_{ki}^3,
\end{aligned}
$$

$$W_{\ell i}W_{ji}^2W_{jq}W_{ki}^3, \quad (W_{\ell i}W_{is}W_{\ell y})(W_{jt}W_{jq}W_{km}W_{kp}),$$
$$W_{\ell i}W_{\ell y}W_{ji}^2W_{jq}W_{km}W_{kp}, \quad W_{\ell i}W_{ji}^2W_{jq}W_{k\ell}^2W_{kp}.$$

For each type, we count m_0, the effective number of indices. It equals to twice of the number of distinct indices in the summand, minus the number of distinct indices appearing in all single-W terms. It turns out that $m_0 \leq 12$ for all types above. By similar arguments as in (272), we conclude that

$$\mathbb{E}[R_{31}^2] \leq \frac{1}{v^6}\cdot C\|\theta\|_1^{m_0} \leq C\|\theta\|_1^{m_0-12} = O(1) = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

Consider R_{33}-R_{34}. We only give the proof when $\|\theta\|^6 \gg [\log(n)]^7$, as it is much simpler. In the case of $1 \ll \|\theta\|^6 \leq C[\log(n)]^7$, we can follow similar steps above to obtain desired bounds, where details are omitted. On the event E (see (271) for definition),

$$\begin{aligned}
|R_{33}| &\leq \sum_{i,j,k,\ell} |\eta_\ell||G_i^2G_j^2G_k^2G_\ell| \\
&\leq C\sum_{i,j,k,\ell}\theta_\ell \frac{\sqrt{\theta_i^2\theta_j^2\theta_k^2\theta_\ell\|\theta\|_1^7[\log(n)]^7}}{(\sqrt{v})^7} \\
&\leq \frac{C[\log(n)]^{7/2}}{\sqrt{\|\theta\|_1^7}}\Big(\sum_i\theta_i\Big)\Big(\sum_j\theta_j\Big)\Big(\sum_k\theta_k\Big)\Big(\sum_\ell\theta_\ell^{3/2}\Big) \\
&\leq \frac{C[\log(n)]^{7/2}}{\sqrt{\|\theta\|_1^7}}\cdot\|\theta\|_1^3\big(\|\theta\|\sqrt{\|\theta\|_1}\big) \\
&\leq C[\log(n)]^{7/2}\|\theta\|,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality $\sum_\ell \theta_\ell^{3/2} \leq \|\theta\|\sqrt{\|\theta\|_1}$ in the second last line. When $\|\theta\|^6 \gg [\log(n)]^7$, the right hand side is $o(\|\theta\|^4)$. Similarly,

$$\begin{aligned}
|R_{34}| &\leq \sum_{i,j,k,\ell}|G_i^2G_j^2G_k^2G_\ell^2| \\
&\leq C\sum_{i,j,k,\ell}\frac{\theta_i\theta_j\theta_k\theta_\ell\|\theta\|_1^4[\log(n)]^4}{v^4} \\
&\leq C[\log(n)]^4.
\end{aligned}$$

When $\|\theta\|^6 \gg [\log(n)]^7$, the right hand side is $o(\|\theta\|^4)$. As we have argued in (271), the event E^c has a negligible effect. It follows that

$$\max\big\{\mathbb{E}[R_{31}^2],\, \mathbb{E}[R_{33}^2],\, \mathbb{E}[R_{34}^2]\big\} = o(\|\theta\|^8), \qquad \text{under both hypotheses.}$$

This is sufficient to guarantee (266)-(267) for R_k.

We have analyzed all 34 terms in Table G.4. The proof is now complete.

G.4.11. *Proof of Lemma G.12.* Consider an arbitrary post-expansion sum of the form

$$\sum_{i,j,k,\ell(dist)} a_{ij}b_{jk}c_{k\ell}d_{\ell i}, \qquad \text{where} \quad a,b,c,d \in \{\widetilde{\Omega}, W, \delta, \tilde{r}, \epsilon\}. \tag{273}$$

Let $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}}, N_\epsilon)$ be the number of each type in the product, where these numbers have to satisfy $N_{\widetilde{\Omega}} + N_W + N_\delta + N_{\tilde{r}} + N_\epsilon = 4$. As discussed in Section G.3, $(Q_n - Q_n^*)$ equals to the sum of all post-expansion sums such that $N_\epsilon > 0$. Recall that

$$\epsilon_{ij} = (\eta_i^*\eta_j^* - \eta_i\eta_j) + (1 - \frac{v}{V})\eta_i\eta_j - (1 - \frac{v}{V})\delta_{ij}.$$

Define

$$\epsilon_{ij}^{(1)} = \eta_i^*\eta_j^* - \eta_i\eta_j, \quad \epsilon_{ij}^{(2)} = (1 - \frac{v}{V})\eta_i\eta_j, \quad \epsilon_{ij}^{(3)} = -(1 - \frac{v}{V})\delta_{ij}.$$

Then, $\epsilon_{ij} = \epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)} + \epsilon_{ij}^{(3)}$. It follows that each post-expansion sum of the form (273) can be further expanded as the sum of terms like

$$\sum_{i,j,k,\ell (dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i}, \quad \text{where } a,b,c,d \in \{\widetilde{\Omega}, W, \delta, \tilde{r}, \epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}\}. \tag{274}$$

Let $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}})$ have the same meaning as before, and let $N_\epsilon^{(m)}$ be the number of $\epsilon^{(m)}$ term in the product, for $m \in \{1, 2, 3\}$. These numbers have to satisfy $N_{\widetilde{\Omega}} + N_W + N_\delta + N_{\tilde{r}} + N_\epsilon^{(1)} + N_\epsilon^{(2)} + N_\epsilon^{(3)} = 4$. Now, $(Q_n - Q_n^*)$ equals to the sum of all post-expansion sums of the form (274) with

$$N_\epsilon^{(1)} + N_\epsilon^{(2)} + N_\epsilon^{(3)} \geq 1. \tag{275}$$

Fix such a post-expansion sum and denote it by Y. We shall bound $|\mathbb{E}[Y]|$ and $\mathrm{Var}(Y)$.

We need some preparation. First, we derive a bound for $|\epsilon_{ij}^{(1)}|$. By definition, $\eta_i = (1/\sqrt{v})\sum_{j\neq i} \Omega_{ij}$ and $\eta_i^* = (1/\sqrt{v_0})\sum_j \Omega_{ij}$. It follows that

$$\eta_i^* = \frac{\sqrt{v}}{\sqrt{v_0}}\eta_i + \frac{1}{\sqrt{v_0}}\Omega_{ii}.$$

We then have

$$\eta_i^*\eta_j^* = \frac{v}{v_0}\eta_i\eta_j + \frac{\sqrt{v}}{v_0}(\eta_i\Omega_{jj} + \eta_j\Omega_{ii}) + \frac{1}{v_0}\Omega_{ii}\Omega_{jj}.$$

Note that $v = \sum_{i\neq j}\Omega_{ij}$ and $v_0 = \sum_{ij}\Omega_{ij} \asymp \|\theta\|_1^2$. It follows that $v_0 - v = \sum_i \Omega_{ii} \leq \sum_i \theta_i^2 \leq \|\theta\|^2$. Therefore,

$$\begin{aligned}
|\eta_i^*\eta_j^* - \eta_i\eta_j| &\leq \Big|1 - \frac{v}{v_0}\Big|\eta_i\eta_j + \frac{\sqrt{v}}{v_0}(\eta_i\Omega_{jj} + \eta_j\Omega_{ii}) + \frac{1}{v_0}\Omega_{ii}\Omega_{jj} \\
&\leq \frac{C\|\theta\|^2}{\|\theta\|_1^2}\cdot\theta_i\theta_j + \frac{C}{\|\theta\|_1}(\theta_i\theta_j^2 + \theta_j\theta_i^2) + \frac{C}{\|\theta\|_1^2}\cdot\theta_i^2\theta_j^2 \\
&\leq C\theta_i\theta_j\cdot\Big(\frac{\|\theta\|^2}{\|\theta\|_1^2} + \frac{\theta_i + \theta_j}{\|\theta\|_1} + \frac{\theta_i\theta_j}{\|\theta\|_1^2}\Big).
\end{aligned}$$

Since $\|\theta\|^2 \leq \theta_{\max}\|\theta\|_1$, the term in the brackets is bounded by $C\theta_{\max}/\|\theta\|_1$. We thus have

$$|\epsilon_{ij}^{(1)}| \leq \frac{C\theta_{\max}}{\|\theta\|_1}\cdot\theta_i\theta_j, \qquad \text{for all } 1 \leq i \neq j \leq n. \tag{276}$$

Second, in Lemmas G.1-G.11, we have studied all post-expansion sums of the form

$$Z \equiv \sum_{i,j,k,\ell (dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i}, \qquad \text{where} \quad a,b,c,d \in \{\widetilde{\Omega}, W, \delta, \tilde{r}\},$$

where $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}})$ are the numbers of each type in the product. We hope to take advantage of these results. Using the proved bounds for $|\mathbb{E}[Z]|$ and $\mathrm{Var}(Z)$, we can get

$$\mathbb{E}[Z^2] \le C(\alpha^2)^{N_{\widetilde{\Omega}}} \cdot f(\theta; N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}}), \tag{277}$$

where $\alpha = |\lambda_2|/\lambda_1$ and $f(\theta; m_1, m_2, m_3, m_4)$ is a function of θ whose form is determined by (m_1, m_2, m_3, m_4). For example,

$$\begin{cases} f(\theta;0,4,0,0) = \|\theta\|^8, & \text{by claims of } X_1 \text{ in Lemmas G.1\&G.3;} \\ f(\theta;4,0,0,0) = \|\theta\|^{16}, & \text{by claims of } X_6 \text{ in Lemma G.3;} \\ f(\theta;3,1,0,0) = \|\theta\|^8\|\theta\|_3^6, & \text{by claims of } X_5 \text{ in Lemma G.3;} \\ f(\theta;1,2,1,0) = \|\theta\|^4\|\theta\|_3^6, & \text{by claims of } Y_2, Y_3 \text{ in Lemma G.5;} \\ f(\theta;1,1,1,1) = \|\theta\|^8, & \text{by claims of } R_9\text{-}R_{11} \text{ in the proof of Lemma G.11.} \end{cases}$$

If there are more than one post-expansion sum that corresponds to the same $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}})$, we use the largest bound to define $f(\theta; N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}})$. Thanks to previous lemmas, we have known the function $f(\theta; m_1, m_2, m_3, m_4)$ for all possible (m_1, m_2, m_3, m_4).

We now show the claim. Recall that Y is the post-expansion sum in (274). The key is to prove the following argument: For any sequence x_n such that $\sqrt{\log(\|\theta\|_1)} \ll x_n \ll \|\theta\|_1$,

$$\begin{aligned} \mathbb{E}[Y^2] \le C(\alpha^2)^{N_{\widetilde{\Omega}}} &\times \Big(\frac{\theta^2_{\max}}{\|\theta\|_1^2}\Big)^{N_\epsilon^{(1)}} \times \Big(\frac{x_n^2}{\|\theta\|_1^2}\Big)^{N_\epsilon^{(2)}+N_\epsilon^{(3)}} \\ &\times f(\theta; m_1, m_2, m_3, m_4)\Big|_{\substack{m_1 = N_{\widetilde{\Omega}} + N_\epsilon^{(1)} + N_\epsilon^{(2)},\ m_2 = N_W, \\ m_3 = N_\delta + N_\epsilon^{(3)},\ m_4 = N_{\tilde{r}},}} \end{aligned} \tag{278}$$

where $(N_{\widetilde{\Omega}}, N_W, N_\delta, N_{\tilde{r}}, N_\epsilon^{(1)}, N_\epsilon^{(2)}, N_\epsilon^{(3)})$ are the same as in (274)-(275), and $f(\theta; m_1, m_2, m_3, m_4)$ is the known function in (277).

We prove (278). Let D be the event

$$D = \{|V - v| \le \|\theta\|_1 x_n\}.$$

In Lemma G.10, we have proved $\mathbb{E}[(Q_n - Q_n^*)^2 \cdot I_{D^c}] = o(1)$. By similar proof, we can show: when $|Y|$ is bounded by a polynomial of V and $\|\theta\|_1$ (which is always the case here),

$$\mathbb{E}[Y^2 \cdot I_{D^c}] = o(1).$$

It follows that

$$\mathbb{E}[Y^2] \le \mathbb{E}[Y^2 \cdot I_D] + o(1). \tag{279}$$

We then bound $\mathbb{E}[Y^2 \cdot I_D]$. In the definition of Y, each $\epsilon^{(2)}$ term introduces a factor of $(1 - \frac{v}{V})$, and each $\epsilon^{(3)}$ term introduces a factor of $-(1 - \frac{v}{V})$. We bring all these factors to the front and re-write the post-expansion sum as

$$Y = (-1)^{N_\epsilon^{(3)}} \Big(1 - \frac{v}{V}\Big)^{N_\epsilon^{(2)} + N_\epsilon^{(3)}} X, \qquad X \equiv \sum_{i,j,k,\ell (dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i}.$$

After the factor $(1 - \frac{v}{V})$ is removed, $\epsilon^{(2)}$ becomes $\eta_i\eta_j$; similarly, $\epsilon^{(3)}$ becomes δ_{ij}. Therefore, in the expression of X,

$$\begin{cases} a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \{\widetilde{\Omega}_{ij}, W_{ij}, \delta_{ij}, \tilde{r}_{ij}, \epsilon_{ij}^{(1)}, \eta_i\eta_j\}, \\ \text{number of } \eta_i\eta_j \text{ in the product is } N_\epsilon^{(2)}, \\ \text{number of } \delta_{ij} \text{ in the product is } N_\delta + N_\epsilon^{(3)}, \\ \text{number of any other term in the product is same as before.} \end{cases} \tag{280}$$

On the event D, $|1 - \frac{v}{V}| \leq \frac{x_n \|\theta\|_1}{C\|\theta\|_1^2} = O(\frac{x_n}{\|\theta\|_1})$. Hence,

$$|Y| \leq C\Big(\frac{x_n}{\|\theta\|_1}\Big)^{N_\epsilon^{(2)}+N_\epsilon^{(3)}} |X|, \qquad \text{on the event } D.$$

It follows that

$$\mathbb{E}[Y^2 \cdot I_D] \leq C\Big(\frac{x_n^2}{\|\theta\|_1^2}\Big)^{N_\epsilon^{(2)}+N_\epsilon^{(3)}} \cdot \mathbb{E}[X^2]. \tag{281}$$

To bound $\mathbb{E}[X^2]$, we compare X and Z. In obtaining (277), the only property of $\widetilde{\Omega}$ we have used is

$$|\widetilde{\Omega}_{ij}| \leq \alpha \cdot C\theta_i\theta_j.$$

In comparison, in the expression of X, we have (by (276) and (81))

$$|\widetilde{\Omega}_{ij}| \leq \alpha \cdot C\theta_i\theta_j, \qquad |\epsilon_{ij}^{(1)}| \leq \frac{\theta_{\max}}{\|\theta\|_1} \cdot C\theta_i\theta_j, \qquad |\eta_i\eta_j| \leq C\theta_i\theta_j. \tag{282}$$

If we consider $(\alpha^{N_{\widetilde{\Omega}}} \cdot (\frac{\theta_{\max}}{\|\theta\|_1})^{N_\epsilon^{(1)}} \cdot 1^{N_\epsilon^{(2)}})^{-1} X$ and $(\alpha^{N_{\widetilde{\Omega}}})^{-1} Z$, we can derive the same upper bound for the second moment of both variables, except that the effective N_δ in X should be $N_\delta + N_\epsilon^{(3)}$ and the effective $N_{\widetilde{\Omega}}$ in X should be $N_{\widetilde{\Omega}} + N_\epsilon^{(1)} + N_\epsilon^{(2)}$. It follows that

$$\begin{aligned} \mathbb{E}[X^2] \leq C(\alpha^2)^{N_{\widetilde{\Omega}}} \times \Big(\frac{\theta_{\max}^2}{\|\theta\|_1^2}\Big)^{N_\epsilon^{(1)}} & \\ \times f(\theta; m_1, m_2, m_3, m_4)\Big|_{\substack{m_1 = N_{\widetilde{\Omega}} + N_\epsilon^{(1)} + N_\epsilon^{(2)},\ m_2 = N_W, \\ m_3 = N_\delta + N_\epsilon^{(3)},\ m_4 = N_{\tilde{r}}.}} & \end{aligned} \tag{283}$$

We plug (283) into (281), and then plug it into (279). It gives (278).

Next, we use (278) to prove the claims of this lemma. Under our assumption, we can choose a sequence x_n such that $\sqrt{\log(\|\theta\|_1)} \ll x_n \ll \|\theta\|_1/\|\theta\|^2$. Also, note that $\|\theta\|_1 \geq \theta_{\max}^{-1}\|\theta\|^2 \gg \|\theta\|^2$. Then,

$$\frac{\theta_{\max}}{\|\theta\|_1} = o(\|\theta\|^{-2}), \qquad \frac{x_n}{\|\theta\|_1} = o(\|\theta\|^{-2}). \tag{284}$$

As a result, since $N_\epsilon^{(1)} + N_\epsilon^{(2)} + N_\epsilon^{(3)} \geq 1$, (278) implies

$$\mathbb{E}[Y^2] = o(\|\theta\|^{-4}) \cdot f(\theta; m_1, m_2, m_3, m_4), \tag{285}$$

for $m_1 = N_{\widetilde{\Omega}} + N_\epsilon^{(1)} + N_\epsilon^{(2)}$, $m_2 = N_W$, $m_3 = N_\delta + N_\epsilon^{(3)}$ and $m_4 = N_{\tilde{r}}$. We then extract $f(\theta; m_1, m_2, m_3, m_4)$ from previous lemmas. Recall the following facts:

- Under the null hypothesis, for any previously analyzed post-expansion sum Z, $|\mathbb{E}[Z]| \leq C\|\theta\|^4$ and $\mathrm{Var}(Z) \leq C\|\theta\|^8$.
- Under the alternative hypothesis, except $\sum_{i,j,k,\ell(dist)} \widetilde{\Omega}_{ij}\widetilde{\Omega}_{jk}\widetilde{\Omega}_{k\ell}\widetilde{\Omega}_{\ell i}$, for all previously analyzed post-expansion sum Z , $|\mathbb{E}[Z]| \leq C\alpha^2\|\theta\|^6$ and $\mathrm{Var}(Z) \leq C\|\theta\|^8 + C\alpha^6\|\theta\|^8\|\theta\|_3^6$.

Therefore, under both hypotheses, except for $(m_1, m_2, m_3, m_4) = (4, 0, 0, 0)$,

$$f(\theta; m_1, m_2, m_3, m_4) \leq C(\|\theta\|^8 + \|\theta\|^{12} + \|\theta\|^8\|\theta\|_3^6) \leq C\|\theta\|^{12}. \tag{286}$$

Consider two cases for Y. The first case is $N_{\widetilde{\Omega}} + N_\epsilon^{(1)} + N_\epsilon^{(2)} \neq 4$. Combining (285)-(286) gives

$$\mathbb{E}[Y^2] = o(\|\theta\|^{-4}) \cdot C\|\theta\|^{12} = o(\|\theta\|^8).$$

The claims follow immediately. The second case is $N_{\widetilde{\Omega}} + N_{\epsilon}^{(1)} + N_{\epsilon}^{(2)} = 4$. In this case,

$$f(\theta; m_1, m_2, m_3, m_4) = f(\theta; 4, 0, 0, 0) = \|\theta\|^{16}.$$

If $N_{\epsilon}^{(1)} + N_{\epsilon}^{(2)} \geq 2$, then by (278) and (284),

$$\mathbb{E}[Y^2] = o(\|\theta\|^{-8}) \cdot C\|\theta\|^{16} = o(\|\theta\|^8).$$

The claims follow. It remains to consider $N_{\epsilon}^{(1)} + N_{\epsilon}^{(2)} = 1$ (and so $N_{\widetilde{\Omega}} = 3$). Write for short $S = 1 - \frac{v}{V}$. By (280),

$$Y = S^{N_{\epsilon}^{(2)}} \cdot X, \qquad \text{where} \quad X = \sum_{i,j,k,\ell (dist)} a_{ij} b_{jk} c_{k\ell} d_{\ell i},$$

and $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ can only take values from $\{\widetilde{\Omega}_{ij}, \epsilon_{ij}^{(1)}, \eta_i \eta_j\}$. So, X is a non-stochastic number. Using (282), we can easily show

$$|X| \leq C\alpha^{N_{\widetilde{\Omega}}} \Big(\frac{\theta_{\max}}{\|\theta\|_1}\Big)^{N_{\epsilon}^{(1)}} \|\theta\|^8.$$

When $(N_{\epsilon}^{(1)}, N_{\epsilon}^{(2)}) = (1, 0)$, we have $Y = X$. By (284), $\frac{\theta_{\max}}{\|\theta\|_1} = o(\|\theta\|^{-2})$. It follows that

$$\mathrm{Var}(Y) = 0, \qquad |\mathbb{E}[Y]| = |X| \leq C\alpha^3 \cdot o(\|\theta\|^{-2}) \cdot \|\theta\|^8 = o(\alpha^4 \|\theta\|^8).$$

This gives the desired claims. When $(N_{\epsilon}^{(1)}, N_{\epsilon}^{(2)}) = (0, 1)$, we have $Y = S \cdot X$. So,

$$|Y| = |X| \cdot |S| \leq C\alpha^3 \|\theta\|^8 \cdot |S|.$$

Note that $S = 1 - \frac{v}{V}$, where $v = \mathbb{E}[V]$. Using the tail bound (254), we can prove $\mathbb{E}[S^2] \leq C\|\theta\|_1^{-2}$. Therefore,

$$\mathbb{E}[Y^2] \leq \frac{C\alpha^6 \|\theta\|^{16}}{\|\theta\|_1^2} \leq C\alpha^6 \|\theta\|^8 \|\theta\|_3^6,$$

where the last inequality is due to $\|\theta\|^4 \leq \|\theta\|_1 \|\theta\|_3^3$ (Cauchy-Schwarz). The claims follow immediately. □

APPENDIX H: ADDITIONAL SIMULATION RESULTS

In Section 5 of the main article, we investigated the numerical performance of SgnT and SgnQ tests and compare them with the EZ and GC tests. Due to space limit, we only reported the sum of the percent of type I errors and the percent of type II errors. It does not show the contribution of each type of errors. We now report separately the percent of each type of errors.

Figures H.1-H.3 here are supplement to Figures 3-5 of the main article, corresponding to Experiments 1-3, respectively. Below is a brief summary of the settings in three experiments:

- *Experiment 1*. In this experiment, $K = 2$, and the degree parameters are iid generated from a uniform distribution (Experiment 1a), a two-point mass (Experiment 1b), and a Pareto distribution (Experiment 1c), respectively.
- *Experiment 2*. In this experiment, K is larger ($K \in \{5, 10\}$) and P is more complicated, and the community sizes are either balanced (Experiment 2a) or unbalanced (Experiment 2b).
- *Experiment 3*. In this experiment, we allow for mixed memberships, where the percent of mixed nodes is 0% (Experiment 3a), 10% (Experiment 3b), and 25% (Experiment 3c), respectively.

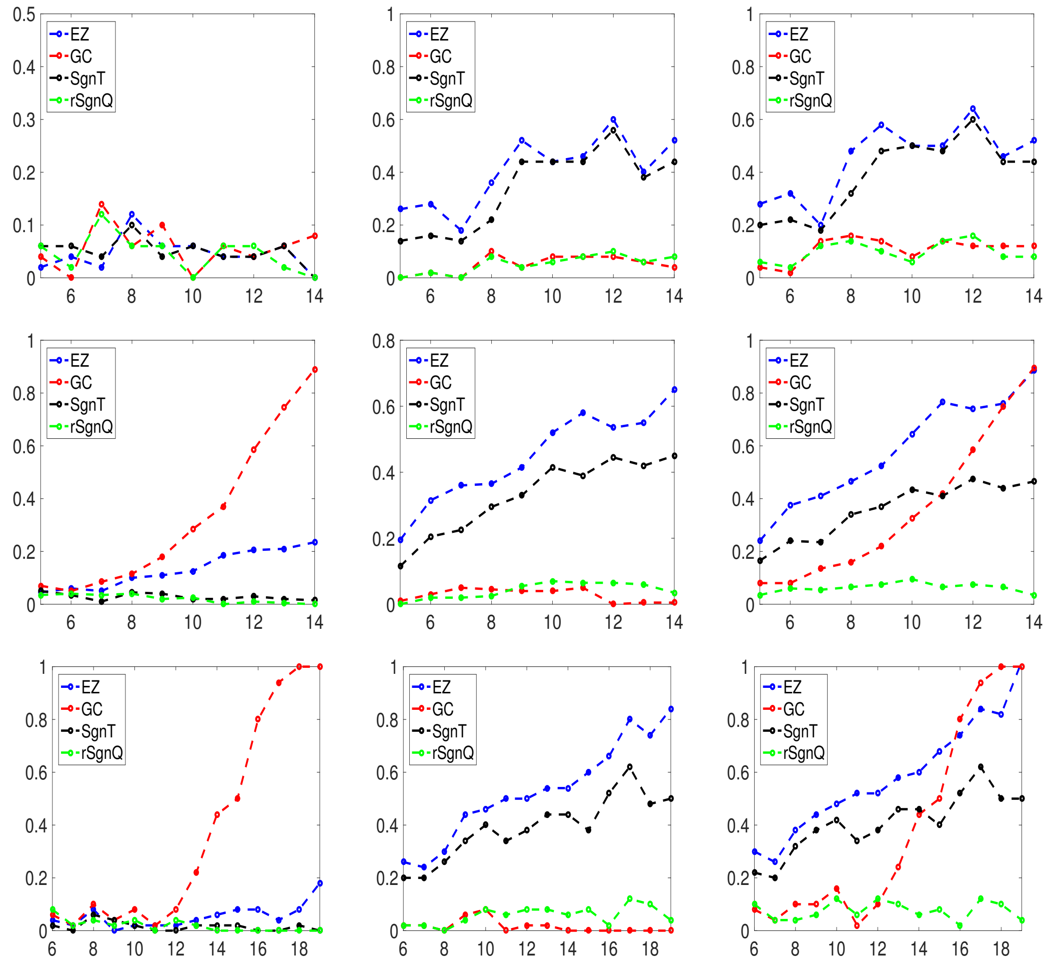

FIG H.1. *Experiment 1 (from top to bottom: Experiment 1a, 1b, and 1c). The x-axis is $\|\theta\|$, and the y-axis is type I error (left), type II error (middle) and the sum (right).*

For each parameter setting, we generate 200 networks under the null hypothesis and 200 networks under the alternative hypothesis, run all the four tests with a target level $\alpha = 5\%$, and record the percent of type I errors, the percent of type II errors, and their sum. In each figure, the plots in the third column are those already shown in the main article.

The results confirm our claims in Section 5. In terms of the type I error, the EZ and GC tests fail to control it at the target level when $\|\theta\|$ is large. It is because the biases of these tests are non-negligible for less sparse networks (the bias of GC is comparably larger). The SgnT and SgnQ tests successfully control the type I error for both sparse and less sparse networks. In terms of the type II error, the order-4 graphlet counting tests have uniformly better power than the order-3 graphlet counting tests. E.g., the type II error of GC is smaller than that of EZ, and the type II error of SgnQ is smaller than that of SgnT.

REFERENCES

[1] BAI, Z. and SILVERSTEIN, J. (2010). *Spectral Analysis of Large Dimensional Random Matrices*. Springer.

[2] HORN, R. and JOHNSON, C. (1985). *Matrix Analysis*. Cambridge University Press.

[3] JIN, J., KE, Z. T. and LUO, S. (2018). Network global testing by counting graphlets. In *Proceedings of the 35th International Conference on Machine Learning, ICML 2018, Stockholmsmässan, Stockholm, Sweden, July 10-15, 2018* 2338–2346.

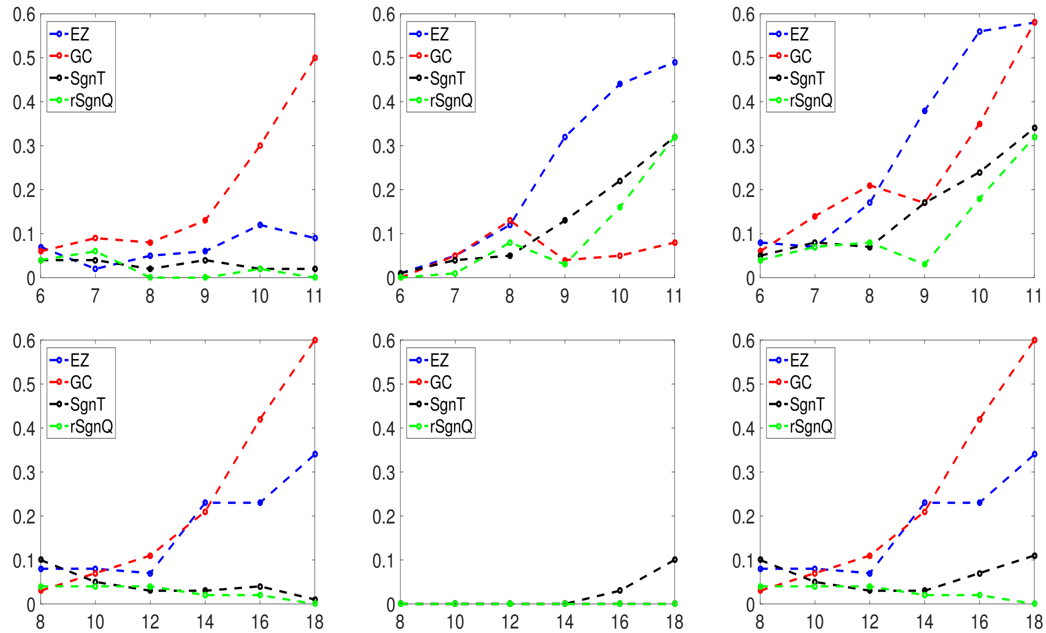

FIG H.2. *Experiment 2 (from top to bottom: Experiment 2a and 2b). The x-axis is $\|\theta\|$, and the y-axis is type I error (left), type II error (middle) and the sum (right).*

[4] JIN, J., KE, Z. T. and LUO, S. (2019). Optimal adaptivity of signed-polygon statistics for network testing. *Manuscript*.

[5] JOHNSON, C. and REAMS, R. (2009). Scaling of symmetric matrices by positive diagonal congruence. *Linear Multilinear A* **57** 123-140.

[6] MARSHALL, A. and OLKIN, I. (1968). Scaling of matrices to achieve specified row and column sums. *Numer. Math.* **12** 83–90.

[7] TROPP, J. (2012). User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.* **12** 389–434.

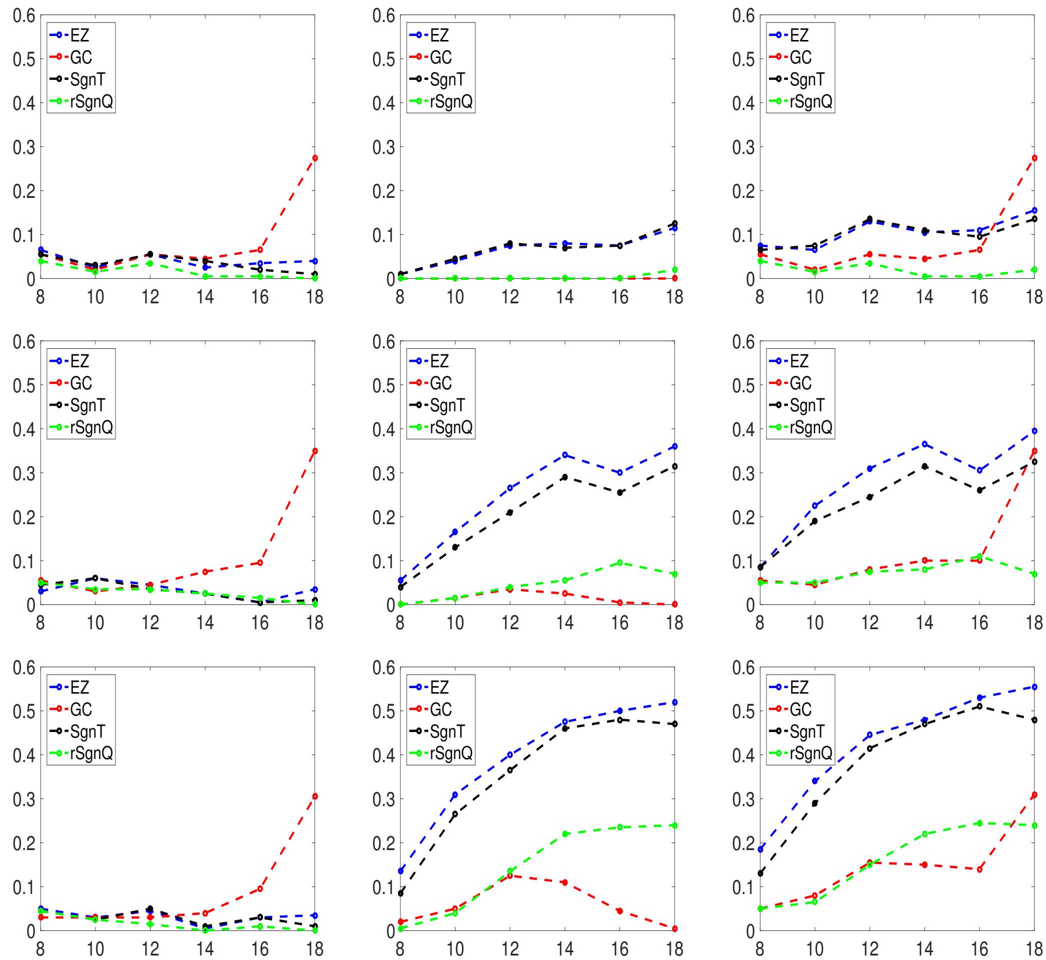

FIG H.3. *Experiment 3 (from top to bottom: Experiment 3a, 3b, and 3c). The x-axis is $\|\theta\|$, and the y-axis is type I error (left), type II error (middle) and the sum (right).*